GENERAL TOPOLOGY

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1. Logic and Set Theory

1.1. **Introduction.** The foundation for the study of each of the main fields of mathematics, topology especially, is set theory. The job of the mathematician is to study sets endowed with certain properties and the consequences that logically result. The mathematician must then also have an understanding of logic! Luckily for everyone involved, the two coincide. That is, set theory is constructed formally with logic and the logic as used by the mathematician can always be rephrased in terms of set theory. The reader is encouraged to then conceptualize however they are most comfortable. Logical proofs are really about contextualized set theory, and set theory is logic in the context of mathematical objects.

General topology, also known as "point-set topology," particularly how the reader will encounter it, can be described by the addition to set theory of only the most general of additional properties. Being a study of mathematics it will be in the course of learning to prove logical statements. With logic giving the very framework of set theory, we have a natural sequence of where to begin the study, *logically*.

1.2. **Logic: Briefly.** The primary concern of logic is the *truth* or *falsity* of *declarative statements*. We are then by principle operating under the *law of excluded middle*: every statement is either true or false, never both. The declarative statements, also referred to as *predicates*, we will denote with capital letters P, Q, etc. If a statement is true or false we will simply denote this with a T or F appropriately.

Two predicates can be related by *connectives*, the truth value of each to be taken as a definition. These truth values can be described by a table that combines all combinations of truth values for both P and Q. The connectives are defined as follows:

(1) AND, denoted $P \wedge Q$: like the English "and," the logical 'and' is true only when both P and Q are simultaneously true.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

(2) OR, denoted $P \vee Q$: The logical OR differs from the English word in order to remove any ambiguity. We say $P \vee Q$ is true whenever P is true, or Q is

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true, or both are true. If one wants to make the declarative statement that of P or Q, but not both, then one must include the "but not both."

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

(3) NOT, denoted $\neg P$: The logical NOT has the opposite truth value of the given predicate. Notice we must have the law of excluded middle to make proper sense of $\neg P$.

P	$\neg P$
T	F
F	T

(4) IF... THEN..., denote $P \Rightarrow Q$: Referred to as a *conditional statement*, one reads the symbols $P \Rightarrow Q$ as "if P, then Q" or as "P implies Q." We refer to P as the *antecedent* (also called the *hypothesis*) and Q as the *conclusion*. In logic, for the connective $P \Rightarrow Q$ to be true, if P is true, then Q must also be true as it is in common speech. However, if P is false, then Q can be either true or false and the connective $P \Rightarrow Q$ will still be true. The logician/mathematician's motivation is thus: A false hypothesis should have no logical implication on the conclusion in general.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

(5) IS EQUIVALENT, $P \iff Q$: Referred to as a *biconditional statement* one reads the symbols $P \iff Q$ as "P is equivalent to Q" or "P if and only if Q". This is a connective of a connective, relating the *if direction*: $Q \Rightarrow P$ (also known as the *converse* of $P \Rightarrow Q$); and the *only if direction*: $P \Rightarrow Q$. It is interpreted in symbolic logic as $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$, thus $P \iff Q$ is true exactly when both P, Q are simultaneously true or false. Mathematical definitions take the form of biconditional statements in order to describe exactly the object of discourse. We illustrate the truth value of $P \iff Q$ with an extended table:

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$P \iff Q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Connectives, by their nature can be combined into larger statements. One such example is $P \wedge (\neg Q)$. When writing compound statements, parenthesis are used as

in arithmetic to determine the order in which connectives are to be considered. We make use of the biconditional to relate equivalent logical statements. For example, $\neg(P \wedge Q) \iff (\neg P) \vee (\neg Q)$. The reader is asked in the exercises to verify this statement using proof tables. A statement such as the previous that is always true is known as a *tautology*. We may also use biconditionals to shorten a connective into a single symbol such as defining R to be the statement such that $R \iff P \vee Q$ is a tautology.

The fundamental goal of any mathematical field is to understand what attributes mathematical objects have that are derived from a primary set of definitions and axioms. One must verify the implicative statement $P \Rightarrow Q$ for two given logical statements P and Q. We want to eliminate the possibility in the truth table of P being true and Q being false a the same time. This is accomplished by a *direct proof*: a sequence of preconfirmed, true conditional and biconditional (definitions) statements that lead from P to Q. As a reductive example, to show $P \Rightarrow Q$, if one knows $P \Rightarrow R$ and that $R \Rightarrow Q$, then one can conclude that $P \Rightarrow Q$. The reader is asked to verify this statement which can be written symbolically as $((P \Rightarrow R) \wedge (R \Rightarrow Q)) \Rightarrow (P \Rightarrow Q)$. This gives a general strategy: as we are assuming P to be true, we recall all of the conditional and biconditional statements that include P as an antecedent. If needed, extend this search to include conditionals and biconditionals with antecedents amongst the conclusions of the previous list. While nowhere near optimized in general, this is approximately the process all mathematicians undertake with experience shortening the search process.

The truth table above defines the conditional to be true whenever the antecedent P is false. It is the case that some statements P are never true. Then $P \Rightarrow Q$ is always true, we refer to a conditional with false antecedent as *vacuously true*.

Example 1.1. Consider the statement about real numbers: If $x^2 < 0$, then $x = 11$. For real numbers x, $x^2 < 0$ is never true thus the conditional is vacuously true.

When attempting to prove a statement $P \Rightarrow Q$, there are occasions when the statement Q is better understood than P. In this case, one may use the *contrapositive*: $(\neg Q) \Rightarrow (\neg P)$. The reader is asked to verify the equivalence of $P \Rightarrow Q$ with the contrapositive.

If the statement $P \Rightarrow Q$ were to be true then $P \wedge (\neg Q)$ must be false. More so, one can verify the tautology $(P \Rightarrow Q) \iff \neg(P \wedge (\neg Q))$. Thus instead of proving $P \Rightarrow Q$, one can prove that $P \wedge (\neg Q)$ is false. This is accomplished by assuming $P \wedge (\neg Q)$ and concluding a that a known false statement is true, typically Q by assumption. Symbolically this reads as $(P \wedge (\neg Q)) \Rightarrow Q$. The statement $(P \wedge (\neg Q)) \Rightarrow Q$, is a *contradiction*. Then it must be that the negation of the antecedent $\neg(P \wedge (\neg Q))$ is true, which is equivalent to $P \Rightarrow Q$. This is known as a *proof by contradiction* and is quite powerful as one now has access to two facts with which to work, P and $\neg Q$. Proofs by contradiction are frequently very similar to proving the contrapositive.

1.3. **Set Theory.** Sets should be intuitively thought of as collections of objects, called *elements*. We know that there are such things as sets as there are objects, namely these notes. Despite the perhaps digital abstract existence of these notes,

there is a set that contains these notes. These notes themselves can be thought of as a set, containing as elements each page. This last thought leads to an important concept for topology, sets can be collections of sets. The set theory as will be utilized here is a part of an axiomatic treatment known as ZFC, the axioms of which we will not distinguish directly but rather discuss as we have need.

We will denote sets by capital letters A, B etc. and elements will be often denoted by lower case letters, a, b, etc. If we wish to say an element belongs to a particular set we write $a \in A$ to mean a is an element of A ($\in$ is a shorthand epsilon, for element). If we wish to specify that b is not in A, we write $b \notin A$. As sets have elements we can refer to them by existence: symbolically we say $\exists a \in A$ to mean "there exists an element a in A". If the existing element is to be unique, we may write $\exists! x$ to mean there exists an x and it is unique. If we wish to remark about all elements of a set A, we symbolize this as $\forall a \in A$ which is read as "for all elements a in A. The symbols $\forall, \exists$ are referred to as *quantifiers*.

Example 1.2. Some common yet important sets we will see in this course are the natural numbers $\mathbb{N}$, the integers $\mathbb{Z}$, the rational numbers $\mathbb{Q}$, and the real numbers $\mathbb{R}$. The properties which makes these sets special will be discussed later, but the reader is assumed to have some general knowledge of these sets.

All discussions of sets and elements comes with a *universal set* U, a set from which we take all of our elements. In many cases the universal set will be one of those discussed in Example 1.2. Sets are completely determined by their elements. This means we consider A and B to be the same set only if they have the same elements. This is written in terms of logic as $\forall u \in U(u \in A \iff u \in B) \Rightarrow (A = B)$.

An element $u \in U$ may have some property which we denote as $P(u)$ and mean to be a logically true statement.

Example 1.3. Let $U = \mathbb{Z}$ be the set of integers, and P the property of being even. Then $P(2)$ is a true statement and $P(3)$ is a false statement.

To describe the elements of a set A, we use curly braces $\{\}$. If A has only a finite number of elements, say a, b, c, then we write $A = \{a, b, c\}$. If we wish to build A by a description based on those elements having a property P, we use *set-builder notation* and write $A = \{x \in U : P(x)\}$. This is read as A is the set of elements of U such that $P(x)$ is true (the colon, sometimes written with a pipe | is always read as 'such that'). There is another special set that we would like to consider, the *empty-set* $\emptyset = \{\}$, defined as the set containing no elements. We include the empty-set as a valid set for discussion due to our construction of sets with logic statements; some statements are never true.

Example 1.4. Let $U = \mathbb{Z}$. If A is the set of all numbers divisible by 2, we may write $A = \{n \in \mathbb{Z} : 2 \text{ divides } n\}$.

Example 1.5. Let $A = \{x \in \mathbb{R} : x^2 < 0\}$. Then $A = \emptyset$. Consider this in the context of our formal definition of equality between sets: $\forall x \in \mathbb{R}(x \in A \iff x \in \emptyset) \Rightarrow (A = \emptyset)$. The conditional $x \in A \Rightarrow x \in \emptyset$ is vacuously true. Likewise the conditionaly $x \in \emptyset \Rightarrow x \in A$ is vacuously true as there are no elements x in $\emptyset$ by definition. Thus we have the equality $A = \emptyset$.

It is the fact that sets are built using logical statements that enables us to combine math and logic.

Example 1.6. Let $A = \{n \in \mathbb{Z} : 2 \text{ divides } n\}$. We make the logical claim that $(n \in A) \Rightarrow (\exists k \in \mathbb{Z} \text{ such that } (n = 2k))$. The proof of this statement follows directly from the definition of what it means for one integer to divide another.

It is often useful (frequently required) to compare to sets. We define A to be a *subset* of B to mean $\forall x \in U\,(x \in A \Rightarrow x \in B)$. If A is a subset of B we write $A \subseteq B$. If A is a subset of B and is not equal to B we say A is a *proper subset* of B, denoted $A \subset B$. Given a universal set U, every set A is a proper subset of U.

Example 1.7. We can rephrase Example 1.6 in terms of subsets. Let $B = \{n \in \mathbb{Z} : \exists k \in \mathbb{Z} \text{ such that } (n = 2k)\}$. Then the conditional in Example 1.6 is true if and only if $A \subseteq B$.

Combining Sets: As we combined logic statements in Section 1, we can combine sets. The definitions of these combinations are each based on logical connectives.

(1) Union: $A \cup B = \{x \in U : (x \in A) \vee (x \in B)\}$.
(2) Intersection: $A \cap B = \{x \in U : (x \in A) \wedge (x \in B)\}$.
(3) Complement: $A^C = \{x \in U : \neg(x \in A)\}$
(4) Difference: $A - B = \{x \in U : (x \in A) \wedge \neg(x \in B)\}$

Given several sets, one can combine them using the above operations in numerous ways. There are certain rules that help us quickly determine the outcome of these combinations. These rules are based entirely on the definitions and the proofs of the validity is left to the reader. Let A, B, C

(1) Distributive Laws:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

(2) *DeMorgan's Laws of sets*:

$$A - (B \cap C) = (A - B) \cup (A - C)$$
$$A - (B \cup C) = (A - B) \cap (A - C)$$

Collections of Sets: Elements of sets are meant to be abstract objects. We frequently have sets of numbers, but the collection of all students enrolled in Dr. Bryant's General Topology course is also a set. Another example is the elements of a set being themselves sets.

Example 1.8. A deck of cards can be considered as a set containing cards. Then the set of all decks of cards is a set of sets.

We typically call a set whose elements are sets a *collection*. One important collection is the *power set*, $\mathscr{P}(A)$, the collection of all subsets of A.

Example 1.9. Let $A = \{a, b\}$. Then $\mathscr{P}(A) = \{\emptyset, \{a\}, \{b\}, A\}$.

We use collections of sets to extend our notions of unions and intersections. Given a collection of sets $\mathscr{A}$, we construct the *arbitrary union* of the elements of $\mathscr{A}$ is defined as

$$\bigcup_{A\in\mathscr{A}} A = \{x \in U : \exists A \in \mathscr{A}(x \in A)\}.$$

Likewise we define the *arbitrary intersection* of the elements of $\mathscr{A}$ as

$$\bigcap_{A\in\mathscr{A}} A = \{x \in U : \forall A \in \mathscr{A}(x \in A)\}.$$

If $\mathscr{A}$ is empty, then applying the definition of $\bigcup A$, no element $x \in U$ satisfies the condition, thus $\bigcup A = \emptyset$. Likewise, applying the definition of the intersection, all $x \in U$ satisfy the condition (vacuously); thus we have $\bigcap A = U$.

Another way to form a new set from old is the *Cartesian product*. This product is defined as the set whose elements are *ordered pairs*. Given sets A and B, the Cartesian product is the set $A \times B$ is the set of ordered pairs (a, b) for which a is an element of A and b is an element of B. Written in set-builder notation, that is:

$$A \times B = \{(a, b) : a \in A \wedge b \in B\}.$$

While ordered pairs themselves can be defined in terms of a collection of sets, we take it to be a primitive definition and recall that 'order' means $(a, b) \neq (b, a)$. Regarding notation, it is unfortunately the case that in the realm of real numbers (a, b) can take on an entirely different meaning, particularly as the set of numbers x such that $a < x < b$. In the event that this could cause confusion we adopt a different notation for the ordered pair (a, b) using the product symbol $a \times b$. Combined with the notation of capital letters for sets, lower-case for elements, $a \times b \in A \times B$, is quite clear.

2. Functions

2.1. **Introduction.** To understand mathematics is to understand functions. In every field of study, functions are used to find similarities between the objects of study. In algebra, these are called homomorphisms; in analysis, functionals; and in topology, we will look at continuous functions. Because of their pervasiveness in the study, you have no doubt encountered functions numerous times. In this section we will look at the formal treatment of functions so that we can rigorously prove theorems concerning their behavior later in the course.

2.2. **Rules and Functions.** Functions are usually encountered in the form of *rules*, assigning to each element of some set A an element of a set B. In calculus we saw these rules written both explicitly:

$$f(x) = x^2 + 1, \quad g(x) = \sin(1/x);$$

and implicitly

$$f(t) = \int_0^t e^{-x^2} dx, \quad \exp(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!}.$$

In the setting of calculus, we also knew what objects were to be input into the function and what to expect as output. We will have need of functions whose inputs are abstract objects. To that end we give precise definitions and descriptive symbolic representations to remove the possibility of confusion.

Definition 2.1. Given two sets C, D and their Cartesian product $C \times D$, a *rule of assignment* is a subset $r \subseteq C \times D$, having the property that each element of C appears as the first coordinate of at most one ordered pair belonging to r.

The above definition is experienced in calculus as the horizontal line test! We can use symbolic logic to rewrite Definition 2.1:

$$((c, d) \in r \wedge (c, d') \in r) \Rightarrow (d = d').$$

When encountering a new definition, the reader should contemplate how the new idea fits with those ideas discussed previously. Here, one could ask, "can a rule assignment every by equal to all of $C \times D$?"

For the rule assignment $r \subseteq C \times D$, the subset of C consisting of all first coordinates of elements of r is the *domain* of r. The subset of D consisting of all second coordinates of elements of r is the *image* of r. In set-builder notation:

$$\text{domain } r = \{c \in C : \exists d \in D((c, d) \in r)\}$$
$$\text{image } r = \{d \in B : \exists c \in C((c, d) \in r)\}.$$

We now have the machinery with which to define functions:

Definition 2.2. A *function* f is a rule assignment $r \subseteq C \times D$, together with a set $B \subseteq D$ containing the image of r. The *domain of* f, $A \subseteq C$ is the domain of r; the *image of* f, denoted by $\text{Im}(f)$ of $f(A)$, is the image of r; and the set B is called the *codomain of* f (the textbook refers to B as the range).

We collect all of this information into a single symbol

$$f : A \to B,$$

and read this as "f is a function from A to B." We will avoid using the word mapping to denote a generic function as mappings have special meaning in topology.

For an element $a \in A$, we say $f(a) \in B$ is the *image of* a or the *value* of a under f. Because f is defined by a rule assignment r, there is a unique element of B such that $(a, f(a)) \in r$. For a subset $A_0 \subseteq A$, we denote the image of all elements $a \in A_0$ as $f(A_0)$.

Definition 2.3. Let $f : A \to B$ be a given function and let A_0 be a subset of A. We define the *restriction of* f *to* A_0 to be the function mapping A_0 into B, whose rule is $\{(a, f(a)) : a \in A_0\}$. We denote the restriction as $f|_{A_0} : A_0 \to B$.

It should be clear from the definition that $\text{Im}(f|_{A_0}) = f(A_0)$ and domain $f|_{A_0} = A_0$, of course. We notice that f and $f|_{A_0}$ are not the same function. They have different domains!

What we have accomplished is a rebuilding of our intuition of functions. Rebuilt with a sure foundation. We can now go back to how we previously communicated functions with rules but with the added benefit of specificity.

Example 2.4. Let f be the function whose rule is $r = \{(x, x^2 + 1) : x \in \mathbb{R}\}$ and the codomain is $\mathbb{R}$. We know the domain of f is $\mathbb{R}$, and the image of f is the interval $[1, \infty) \subset R$. If we want to use symbols we may write $f : \mathbb{R} \to \mathbb{R}$ is the function such that $f(x) = x^2 + 1$ and we record all of the same information compactly.

Example 2.5. Let $g : \mathbb{Z} \to \mathbb{Z}$ given by $g(x) = x^2 + 1$. The function g is not the same function as f in Example 2.4, how can we tell? Because $\frac{16}{9} \in \text{Im}(f)$, but $\frac{16}{9} \notin \text{Im}(g)$. Returning to the definition of set equality, $\text{Im}(g) \neq \text{Im}(f)$. Functions are meant to include all of the associated information in a single object: domain, image, codomain, rule, all together.

We see from Examples 2.4 and 2.5 that simply writing $f(x) = x^2 + 1$ is no longer sufficient to specify a function, we must communicate the domain and codomain with the rule.

Definition 2.6. Given functions $f : A \to B$ and $g : B \to C$, we define the *composite* $g \circ f$ of f and g as $g \circ f : A \to C$ defined by the rule $(g \circ f)(a) = g(f(a))$.

The composition $g \circ f$ is only defined when the codomain of f is equal to the domain of g. Composition can be thought of as moving the point a to $f(a)$ in B, then to the point $g(f(a))$ in C. This is illustrated in Figure 1.

Example 2.7. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$ and $g : \bar{\mathbb{R}}_+ \to \mathbb{R}$ be given by $g(x) = \sqrt{x}$ (here $\bar{\mathbb{R}}_+$ is the set of non-negative real numbers). The composite function $f \circ g : \bar{\mathbb{R}}_+ \to \mathbb{R}$ is given by the rule $(f \circ g)(x) = x$. We cannot construct the composite $g \circ f$ however, as the domain of g is not the codomain of f. We are able to construct a new function $f^* : \mathbb{R} \to \bar{\mathbb{R}}_+$ by the same rule, $f^*(x) = x^2$. Then $g \circ f^* : \mathbb{R} \to \mathbb{R}$ is defined and given by the rule $(g \circ f^*)(x) = |x|$.

We now look at some basic but fundamental properties of functions.

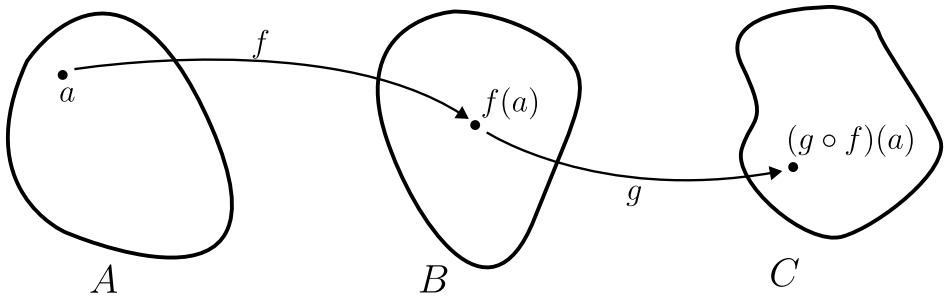


FIGURE 1. Composition of $f : A \to B$ and $g : B \to C$.

Definition 2.8. Let $f : A \to B$ be given. The function f is said to be *injective* (*one-to-one*) if for every pair of points in A, their values are distinct. This is equivallent to the following statement being true:

$$\forall a \in A \, \forall a' \in A \, [(f(a) = f(a')) \Rightarrow (a = a')].$$

The function f is said to be *surjective* (*onto*) if every element of B is the image of some element of A. This is equivalent to the following statement being true:

$$\forall b \in B \, \exists a \in A \, (b = f(a))$$

If f is both injective and surjective, then we say f is *bijective*.

The definition of injectivity depends only on the domain of f, while surjectivity depends on both the domain and codomain. In the homework you will be asked to show that injectivity and surjectivity are preserved under compositions. It follows that bijectivity is preserved under composition.

Example 2.9. Let f, f^*, g be the functions given in Example 2.7. Then f is neither injective or surjective; f^* is surjective but not injective; g is injective but not surjective.

Definition 2.10. Let $f : A \to B$ be given and let B_0 be a subset of B. We define the *preimage of* B_0 *under* f to be the set

$$f^{-1}(B_0) = \{a \in A : f(a) \in B_0\}.$$

For a subset A_0 of A it is not generally true that $f^{-1}(f(A_0)) = A_0$ (read the preimage of the image of A_0) nor is it generally true that $f(f^{-1}(B_0)) = B_0$ (read the image of the preimage of B_0). The relevant rules, which can be verified via element chasing, are:

$$A_0 \subseteq f^{-1}(f(A_0))$$
$$f(f^{-1}(B_0)) \subseteq B_0$$

The first inclusion is equality if f is injective, while the second inclusion is equality if f is surjective.

Example 2.11. Again consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$, and consider the interval subset of the domain $[0, 1]$,

$$[0, 1] \subset f^{-1}(f([0, 1])) = f^{-1}([0, 1]) = [-1, 1]$$

Similarly consider the subset of the codomain $[-4, 4]$,

$$f(f^{-1}([-4, 4])) = f([0, 2]) = [0, 4] \subset [-4, 4]$$

This leads us to the following observation:

Definition 2.12. If $f : A \to B$ is bijective function, then for all $b \in B$ the preimage $f^{-1}(b)$ is a unique element $f^{-1}(b) = a \in A$. If f has the rule assignment r, then the subset $r^* \subseteq B \times A$ given by $r^* = \{(b, f^{-1}(b)) : (f^{-1}(b), b) \in r\}$ is a rule assignment. We call the corresponding function the *inverse* of f and denote it by $f^{-1} : B \to A$.

When $f : A \to B$ is bijective and $B_0 \subseteq B$, we have two meanings for $f^{-1}(B_0)$: the preimage of B_0 under f, and the image of B_0 under the inverse function $f^{-1} : B \to A$. The reader can check that these two meanings give the same set. Hence, there is no ambiguity.

The following lemma shows that the inverse function is unique, it further gives a criterion for checking if a function is a bijection.

Lemma 2.13. *Let $f : A \to B$ be given. If there exist functions $g, h : B \to A$ such that $g(f(a)) = a$ for all $a \in A$ and $f(h(b)) = b$ for all $b \in B$, then f is bijective. Further if f^{-1} is the inverse of f, then $g = h = f^{-1}$.*

Example 2.14. Let $f : \mathbb{R} \to \mathbb{R}_+$ be given by $f(x) = e^x$ and let $g : \mathbb{R}_+ \to \mathbb{R}$ be given by $g(x) = \ln(x)$. We can conclude that f is bijective sense for all $x \in \mathbb{R}$, $g(f(x)) = x$ and for all $y \in \mathbb{R}_+$, $f(g(y)) = y$.

3. Relations

3.1. **Introduction.** In this section we will look at the abstract notion of relations through two key examples: order relations and equivalence relations. We will make extensive use of order relations throughout the course in the construction of topological spaces. Equivalence relations (as the name suggests) is our way of assigning equality to abstract objects. Later in the course we will define what it means for two topological spaces to be the same; this will be done through an equivalence relation.

3.2. **Order Relations.** One way to conceptualize relations is as an abstraction of rule assignments, which are subsets $r \subseteq A \times B$.

Definition 3.1. A *relation* on a set A is a subset $C \subseteq A \times A$. If $(x, y) \in C$ we write xCy and read this as "x is in relation C to y".

Then a function $f : A \to A$ is a special type of relation on A. Notice that every subset of $A \times A$ is a relation on A but not necessarily a rule assignment. In this way, the definition of relations does not give very much structure. We turn our attention then to order relations:

Definition 3.2. A relation C on a set A is called a *strict total order* (the textbook refers to this as an *order relation*) if it has the following properties:

(1) (Comparability) For every $x, y \in A$ for which $x \neq y$, either xCy or yCx.
(2) (Nonreflexivity) For no x in A is xCx.
(3) (Transitivity) If xCy and yCz, then xCz.

Relations satisfying (1) and (3) are called total orders, those satisfying only (2) and (3) are called strict partial orders, and so the naming conventions go. When C is a strict total order we typically use the "less than" symbol in place of the character C; we write $x < y$ to mean xCy. We shall use the symbol $x \leq y$ to mean $x < y$ or $x = y$. If we want to position y first, we can write $y > x$ to also mean xCy. We will find it convenient to combine $x < z \wedge z < y$ into a 'string' of inequalities $x < z < y$ (with obvious generalizations to the symbols $>, \leq, \geq$).

Example 3.3. Let A be a set with total order $<_A$. Let A_0 be a subset of A. The relation $<_{A_0}$ defined by

$$a_0 <_{A_0} a_1 \Rightarrow a_0 <_A a_1$$

is a total order. The proof is left to the reader.

Definition 3.4. Let $<$ be a strict total order on a set X. If $a < b$ we denote the *open interval* as

$$(a, b) = \{x \in X : a < x < b\},$$

the *closed interval* as

$$[a, b] = \{x \in X : a \leq x \leq b\},$$

and the *half-open intervals* as

$$[a, b) = \{x \in X : a \leq x < b\}$$
$$(a, b] = \{x \in X : a < x \leq b\}$$

Suppose A is a set with strict total order $<$. Let A_0 be a subset of A. We say that A_0 is *bounded above* (*bounded below*) if $\exists b \in A$ such that $\forall a \in A_0$ we have $a \leq b$ ($a \geq b$), b is called an *upper bound or* (*lower bound*) for A_0 and if $b \in A_0$ we say b is the *largest element* (*smallest element*) of A_0.

If the set of all upper bounds of A_0 has a smallest element, we call it the *least upper bound*; if the set of all lower bounds has a largest element, we call it the *greatest lower bound*. In general for a set A and strict total order $<$ it is not the case that a every subset $A_0 \subseteq A$ that is bounded above (below) has a least upper (greatest lower) bound.

Definition 3.5. A set A with a strict total order $<$ has the *least upper bound property* if every nonempty subset $A_0 \subseteq A$ that is bounded above has a least upper bound. A is said to have the *greatest lower bound property* if every nonempty subset $A_0 \subseteq A$ that is bounded below has a greatest lower bound.

In the homework you will show that if A has the least upper bound property then it has the greatest lower bound property.

Definition 3.6. Let A and B be two sets with strict total orders $<_A$ and $<_B$ respectively. Define the relation $<$ on $A \times B$ by defining

$$a_0 \times b_0 < a_1 \times b_1 \text{ if } \begin{cases} a_0 <_A a_1, \text{ or} \\ a_0 = a_1 \wedge b_0 <_B b_1 \end{cases}$$

Then $<$ is a strict total order called the *dictionary order* on $A \times B$.

Note that $<$ is a subset of the Cartesian product $(A \times B) \times (A \times B)$. The dictionary order can be extended to a finite product $A_1 \times A_2 \times \cdots \times A_n$.

We now have our first use of functions to measure structure:

Definition 3.7. Let A and B be two sets with order relations $<_A$ and $<_B$ respectively. We say A has the same *order type* as B if there is exists a bijective function $f : A \to B$ such that

$$\forall a_1, a_2 \in A \quad a_1 <_A a_2 \iff f(a_1) <_B f(a_2)$$

An important example of a set with a strict total order is the real numbers. In this course, we may take the real numbers to be a set $\mathbb{R}$ with the usual operations of addition and multiplication satisfying the typical algebraic properties and a strict total order $<$ that satisfies the:

Mixed Algebraic and Order Properties

(1) $\forall x, y \in \mathbb{R}$ if $x < y$, then $x + z < y + z$.
(2) $\forall x, y \in \mathbb{R}$ if $x < y$ and $z > 0$, then $xz < yz$.

Order Properties

(3) The order relation $<$ has the least upper bound property.
(4) (completeness) $\forall x, y \in \mathbb{R}$ if $x < y$, then $\exists z$ such that $x < z < y$.

This order is quite special in the sense that any set of numbers with these algebraic properties with a strict total order satisfying the order properties and mixed algebraic properties is of the same order type as $\mathbb{R}$.

We can recover the integers from this definition of the real numbers

Definition 3.8. A subset A of the real numbers is said to be inductive if it contains the number $1 \in \mathbb{R}$, and if $\forall x \in \mathbb{R}\,(x \in A \Rightarrow x + 1 \in A)$. Let $\mathscr{A}$ be the collection of all inductive subsets of $\mathbb{R}$. Then the set $\mathbb{Z}_+$ of *positive integers* is defined by the arbitrary intersection

$$\mathbb{Z}_+ = \bigcap_{A \in \mathscr{A}} A$$

We define the integers to be $\mathbb{Z} = \mathbb{Z}_+ \cup \{0\} \cup \{x \in \mathbb{R} : -x \in \mathbb{Z}_+\}$, the positive integers, their negatives, and 0.

Example 3.9. The order types of $\mathbb{Z}$ and $\mathbb{R}$ have different order types. See the section on countable and uncountable sets.

Example 3.10. The interval $(0, 1) \subset \mathbb{R}$ has the same order type as $\mathbb{R}$. Consider the order preserving function $\sigma : \mathbb{R} \to (0, 1)$ given by $\sigma(x) = \frac{1}{1+e^{-x}}$. It is left to the reader to show that σ is bijective and order preserving.

Example 3.11. The set $A = \mathbb{Z}_+ \times [0, 1)$ with the dictionary order, where $[0, 1)$ and $\mathbb{Z}_+$ have the order induced as subsets of $\mathbb{R}$. The function $f : A \to \mathbb{R}$ given by $f(n, x) = n + x - 1$ is an order preserving bijection. In contrast $B = \mathbb{Z}_+ \times [0, 1]$ with the dictionary order is not of the same order type. This is easily verified as the order on A is complete, that is for all $n_0 \times x_0, n_1 \times x_1 \in A$ there is an element $m \times y \in A$ such that $n_0 \times x_0 < m \times y < n_1 \times x_1$. However the elements 1×1 and 2×0 are distinct elements of B with no intermediate elements. That is B is not complete under the dictionary order.

3.3. **Equivalence Relations.** We now look at the may mathematicians discuss sameness of objects: through a certain type of relation.

Definition 3.12. A relation C on a set A is called an *equivalence relation* if it has the following properties:

(1) (Reflexivity) For all $x \in A$, xCx.
(2) (Symmetry) If xCy, then yCx.
(3) (Transitivity) If xCy and yCz, then xCz.

There are several symbols we often see for equivalence relations: $=$, $\sim$ (\sim in LaTeX), $\simeq$ (\simeq), $\cong$ (\cong), $\equiv$ (\equiv), $\approx$ (\approx). We will follow the Munkres in using $\sim$ for a general equivalence relation.

Example 3.13. Let $A = \mathbb{Z}$, and define the relation $\equiv$ on A by

$$n \equiv m \text{ if } \exists k \in \mathbb{Z} \text{ such that } n - m = 2k.$$

Let us show that $\equiv$ is an equivalence relation on A

Proof. We check the three properties in the definition of equivalence relations:

(1) (Reflexivity) Let $n \in A$ be given. As $n - n = 0 = 2(0)$, $n \equiv n$. Hence, the relation is reflexive.
(2) (Symmetry) Suppose $n \equiv m$. By supposition there exists $k \in \mathbb{Z}$ such that $n - m = 2k$. We then compute $m - n = -(n - m) = 2(-k)$. Thus $m \equiv n$, and we conclude the relation is symmetric.

(3) (Transitivity) Suppose $n \equiv m$ and $m \equiv p$. Then there exists $k, l \in \mathbb{Z}$ such that $n - m = 2k$ and $m - p = 2l$. We compute $n - p = (n - m) + (m - p) = 2k + 2l = 2(k + l)$. Thus the relation is transitive.

We have shown the relation $\equiv$ is reflexive, symmetric, and transitive, and hence an equivalence relation on the set A. □

Example 3.14. Check that our definition of equality of sets is an equivalence relation.

Given an equivalence relation $\sim$ on A and an element $a \in A$ we define a subset $E \subseteq A$, called the *equivalence class* of a, by

$$E = \{x \in A : x \sim a\}.$$

Note that E is always nonempty as $a \in E$.

Lemma 3.15. *Let A be a set and let $\sim$ be an equivalence relation on A. Two equivalence classes E and E' are either equal as sets or disjoint.*

Proof. Let A and $\sim$ be given. Let E be equivalence class determined by $a \in A$ and E' be the equivalence class determined by $a' \in A$. If $E \cap E' = \emptyset$ we are done. So, suppose that $\exists b \in E \cap E'$. Then $a \sim b$ and $a' \sim b$. By transitivity, $a \sim a'$. Suppose $c \in E$. Again by the transitive property applied to $c \sim a$ and $a \sim a'$, $c \sim a' \Rightarrow c \in E'$. Thus $E \subseteq E'$. By a symmetric argument we can also conclude $E' \subseteq E$ giving equality. Thus E and E' are either equal or disjoint, completing the proof.

□

Definition 3.16. Let A be a set. A *partition* of A is a collection $\mathscr{C}$ of subsets of A such that for all distinct pairs $E, E' \in \mathscr{C}$, $E \cap E' = \emptyset$ and $\bigcup_{E \in \mathscr{C}} E = A$.

By Lemma 3.15, every equivalence relation defines a partition. The reader is encouraged to verify this fact along with the claim that every partition defines and equivalence relation.

Example 3.17. Let $A = \mathbb{Z}$. Define $E = \{n \in A : \exists k \in \mathbb{Z}(n = 2k)\}$ and $E' = \{n \in A : \exists k \in \mathbb{Z}(n = 2k + 1)\}$. Check that $\mathscr{C} = \{E, E'\}$ is a partition of A. How does the equivalence relation associated to $\mathscr{C}$ compare to that given in Example 3.13?

4. Topological Spaces

4.1. **Introduction.** The study of topological spaces can be traced back to Leibniz and the goal of understanding Euclidean space, subsets of Euclidean space, and continuous functions on Euclidean space. As the generations that succeeded him continued to add to what was known about these objects, topology was born. Despite, at the time, how little was understood about Euclidean space (at least rigorously), the foundations for topology as its own field were solid.

Creating an axiomatic field is no meager task. If the definitions are too specific, you may not have varied enough objects to be of interest; if the definitions are too abstract, then there will not be enough structure to prove any fundamental results. With the origins tied to analysis of the Euclidean plane, the optimal outcome is to have as general of a definition as possible while still being able to prove those theorems the zeitgeist determines as essential for abstract topological spaces. Let us now familiarize ourselves with these fundamental definitions.

4.2. **Topologies.**

Definition 4.1. Let X be a set. A collection $\mathscr{T}$ of subsets of X is called a *topology* on X if $\mathscr{T}$ has the following properties:

i) $\emptyset \in \mathscr{T}$ and $X \in \mathscr{T}$.
ii) If $\{U_1, U_2, \ldots, U_n\} \subseteq \mathscr{T}$ is a finite subcollection of $\mathscr{T}$ then $\bigcap_{i=1}^{n} U_i \in \mathscr{T}$.
iii) If $\mathscr{U} \subseteq \mathscr{T}$ is any subcollection of $\mathscr{T}$, then $\bigcup_{U \in \mathscr{U}} U \in \mathscr{T}$.

When X and a topology $\mathscr{T}$ are specified, the pair $(X, \mathscr{T})$ is called a *topological space*.

To understand this definition better, we say that a topology is a collection of subsets that contains both the empty and whole set, is closed under finite intersections, and is closed under arbitrary unions. The subsets U of X that are elements of $\mathscr{T}$ are called *open sets*.

While the choices for the axioms may seem random, that these properties hold for certain subsets of Euclidean space and are foundational to our understanding of Euclidean space should become clear in future sections. Further, if the roles of ii) and iii) are reversed, one takes finite unions and arbitrary intersections, the resulting theory will be the same. The reader is encouraged to come back to this remark in the future with a proof in hand.

It is hopefully clear that for a given set X there are many topologies.

Example 4.2. Let $X = \{a, b, c\}$. The collections $\mathscr{T}_1 = \{\emptyset, X, \{a\}\}$ and $\mathscr{T}_2 = \{\emptyset, X, \{b, c\}\}$ are both topologies on X.

Example 4.3. Let X be a set. The collection of all subsets of X (that is $\mathscr{P}(X)$) is a topology on X called the *discrete topology*. The collection $\mathscr{T} = \{\emptyset, X\}$ is a topology on X called the *indiscrete topology* or the *trivial topology*.

Example 4.4. Let X be a set. Define $\mathscr{T}_f$ and $\mathscr{T}_c$ by

$$\mathscr{T}_f = \{U \subset X : X - U \text{ is finite}\} \cup \{\emptyset, X\}$$
$$\mathscr{T}_c = \{U \subset X : X - U \text{ is countable}\} \cup \{\emptyset, X\}$$

Then $\mathscr{T}_f$ is a topology on X called the *cofinite topology* and $\mathscr{T}_c$ is a topology on X called the *cocountable topology*.

Lemma 4.5. *Let X be a set and let $\mathscr{A} = \{\mathscr{T}_\alpha\}$ be a collection of topologies on X. Then $\bigcap_{\mathscr{T}_\alpha \in \mathscr{A}} \mathscr{T}_\alpha$ is a topology on X.*

Proof. Let X be a set and let $\mathscr{A} = \{\mathscr{T}_\alpha\}$ be a collection of topologies on X. Let us write $\mathscr{F} = \bigcap_{\mathscr{T}_\alpha \in \mathscr{A}} \mathscr{T}_\alpha$. We first note that as $U \in \mathscr{F}$ implies $U \in \mathscr{T}_\alpha$ for all $\mathscr{T}_\alpha \in \mathscr{A}$, the elements of $\mathscr{T}_\alpha$ are subsets of X. As $\emptyset \in \mathscr{T}_\alpha$ and $X \in \mathscr{T}_\alpha$ for all $\mathscr{T}_\alpha \in \mathscr{A}$, then we have $\emptyset, X \in \mathscr{F}$.

Let $\{U_1, U_2, \ldots, U_n\}$ be a finite subcollection of $\mathscr{F}$. Then for all $\mathscr{T}_\alpha$, $\{U_1, U_2, \ldots, U_n\} \subseteq \mathscr{T}_\alpha$. As $\mathscr{T}_\alpha$ is a topology on X, we have that $\bigcap_{i=1}^n U_i \in \mathscr{T}_\alpha$. Thus $\bigcap_{i=1}^n U_i \in \mathscr{F}$. As $\{U_1, U_2, \ldots, U_n\}$ was arbitrary, $\mathscr{F}$ is closed under finite intersections.

Let $\mathscr{U} \subseteq \mathscr{F}$ be an arbitrary subcollection of $\mathscr{F}$. Then for all $\mathscr{T}_\alpha$, $\mathscr{U} \subseteq \mathscr{T}_\alpha$. As $\mathscr{T}_\alpha$ is a topology on X, we have that $\bigcup_{U \in \mathscr{U}} U \in \mathscr{T}_\alpha$. Thus $\bigcup_{U \in \mathscr{U}} U \in \mathscr{F}$. As $\mathscr{U}$ was arbitrary, $\mathscr{F}$ is closed under arbitrary unions.

We have shown that $\mathscr{F}$ contains the empty set and the set X, is closed under finite intersections and arbitrary unions of subsets of X. Hence, $\mathscr{F}$ is a topology on X. □

Does Lemma 4.5 hold for unions of topologies??

While the structure of topologies on a given set X can vary in the extreme, we have a way to compare topologies.

Definition 4.6. Suppose $\mathscr{T}$ and $\mathscr{T}'$ are topologies on some set X. Suppose that $\mathscr{T} \subseteq \mathscr{T}'$. We say that the topology $\mathscr{T}'$ is *finer* than $\mathscr{T}$; that $\mathscr{T}$ is *coarser* than $\mathscr{T}'$. We say $\mathscr{T}$ and $\mathscr{T}'$ are comparable if either $\mathscr{T} \subseteq \mathscr{T}'$ or $\mathscr{T}' \subseteq \mathscr{T}$.

We notice immediately that the the discrete topology is both comparable to and finer than every other topology on some X. Likewise, the indiscrete topology is coarser than every other topology on X. Certainly not every pair of topologies are comparable; consider $\mathscr{T}_1$ and $\mathscr{T}_2$ in Example 4.2.

While Definition 4.1 is exactly the one we want for topological spaces. It is not immediately clear how to describe a general topology on X. That is to say, writing down all of the open sets is not feasible and using set builder notation as in Examples 4.4 may not be tractable in general. In the next section we will address this issue by developing a means to generate a topology from simple descriptions.

5. Bases

5.1. **Introduction.** In the previous section, we defined the notion of a topology on a set. We specified topologies by listing out the open sets, or describing them using set-builder notation. In general, it can be difficult to describe the topology and difficult to utilize these descriptions. For instance, the definitions of cofinite and cocountable topologies is not constructive; can you describe every open set in the cocountable topology on $\mathbb{R}$? In this section we give two means by which we can build a topology from a (hopefully simple) collection of subsets.

5.2. **Basis and Subbasis.**

Definition 5.1. Let X be a set. A collection $\mathscr{B}$ of subsets of X is called a *basis* for a topology on X if $\mathscr{B}$ has the following properties:

i) $\forall x \in X\ \exists B \in \mathscr{B}$ such that $x \in B$.
ii) Let $B_1, B_2 \in \mathscr{B}$. If $x \in B_1 \cap B_2$ then $\exists B_3 \in \mathscr{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

We call the elements of $\mathscr{B}$ *basis elements.*

Definition 5.2. Let X be a set and let $\mathscr{B}$ be a basis. We define the *topology generated by* $\mathscr{B}$ as the collection

$$\mathscr{T} = \{U \subseteq X : (x \in U) \Rightarrow (\exists B \in \mathscr{B}(x \in B \subseteq U))\}$$

Let us first dissect Definition 5.2. We can restate the definition of $\mathscr{T}$ by saying a subset U of X is open if for each element x of U there is a basis element containing x that is a subset of U. The basis elements satisfy this criterion by default; with the respect to $\mathscr{T}$, we can refer to the basis elements as *basic open sets.* But we are getting ahead of ourselves. To call $U \in \mathscr{T}$ and $B \in \mathscr{B}$ open, we must show that $\mathscr{T}$ is actually a topology.

Lemma 5.3. *Let X be a set and let $\mathscr{B}$ be a basis. The collection*

$$\mathscr{T} = \{U \subseteq X : (x \in U) \Rightarrow (\exists B \in \mathscr{B}(x \in B \subseteq U))\}$$

is a topology on X.

Proof. Let $X, \mathscr{B}$ be given and let $\mathscr{T}$ be defined as above. We first show that $\emptyset, X \in \mathscr{T}$.

Consider $\emptyset \subset X$. The statement $x \in \emptyset \Rightarrow (\exists B \in \mathscr{B}(x \in B \subseteq \emptyset))$ is vacuously true as there are no x in $\emptyset$. Thus $\emptyset \in \mathscr{T}$. Considering the whole set X, by Definition 5.1 part i), for all $x \in X$ there is a $B \in \mathscr{B}$ such that $x \in B$. Hence, $X \in \mathscr{T}$.

Let $\{U_1, U_2, \ldots, U_n\}$ be a finite subcollection of $\mathscr{T}$, and let $x \in \bigcap_{i=1}^n U_i$. As each $U_i \in \mathscr{T}$, there exists by definition a $B_i \in \mathscr{B}$ such that $x \in B_i \subseteq U_i$. By induction on Definition 5.1 part ii) there exists a $B \in \mathscr{B}$ such that $x \in B \subseteq \bigcap_{i=1}^n B_i$. We have then constructed the basis element such that $x \in B \subseteq \bigcap_{i=1}^n U_i$. Thus we have $\bigcap_{i=1}^n U_i \in \mathscr{T}$.

Let $\mathscr{U} \subseteq \mathscr{T}$ be a subcollection and let $x \in \bigcup_{U \in \mathscr{U}} U$. That is for some $U \in \mathscr{U} \subseteq \mathscr{T}$, $x \in U$. As $U \in \mathscr{T}$, there is a $B \in \mathscr{B}$ such that $x \in B \subseteq U$. Then $x \in B \subseteq \bigcup_{U \in \mathscr{U}} U$, implying $\bigcup_{U \in \mathscr{U}} U \in \mathscr{T}$.

We have shown that $\mathscr{T}$ contains the empty-set and X, is closed under finite intersections, and is closed under arbitrary unions. We can now conclude that $\mathscr{T}$ is a topology on X. □

Example 5.4. Let X be a set. The collection of all singleton subsets of X is a basis. The topology generated by this collection is the discrete topology.

Example 5.5. Let X be a set and let $\mathscr{B} = \{X\}$. Then $\mathscr{B}$ is a basis for the indiscrete topology.

Definition 5.2 tells us how to describe the topology generated by a given basis. There is another description for $\mathscr{T}$.

Lemma 5.6. *Let X be a set and let $\mathscr{B}$ be a basis for the topology $\mathscr{T}$. Then*

$$\mathscr{T} = \left\{ U \subseteq X : \exists \mathscr{B}' \subseteq \mathscr{B} \text{ such that } U = \bigcup_{B \in \mathscr{B}'} B \right\}.$$

Proof. Let X and $\mathscr{B}$ and let $\mathscr{T}$ be the topology on X generated by $\mathscr{B}$. Let $\mathscr{T}' = \{U \subseteq X : \exists \mathscr{B}' \subseteq \mathscr{B} \text{ such that } U = \bigcup_{B \in \mathscr{B}'} B\}$. Suppose $U \in \mathscr{T}'$. Then $U = \bigcup_{B \in \mathscr{B}'} B$, and as each B is open in $\mathscr{T}$, $U = \bigcup_{B \in \mathscr{B}'} B \in \mathscr{T}$. Hence $\mathscr{T}' \subseteq \mathscr{T}$.

Let $U \in \mathscr{T}$. By definition, for each $x \in U$, there exists $B_x \in \mathscr{B}$ such that $x \in B_x \subseteq U$. Then $U = \bigcup_{x \in U} B_x$ is the union of basis elements. Thus $U \in \mathscr{T}'$.

We have then shown that the topology on X generated by $\mathscr{B}$ is the collection of all unions of basic open sets. □

Between Definition 5.2 and Lemma 5.6 we have two ways to describe the topology associated to a given basis. It is often that we need to perform the opposite task: determine if a collection of subsets is a basis for a given topology.

Lemma 5.7. *Let X be a topological space with topology $\mathscr{T}$. Suppose $\mathscr{C} \subseteq \mathscr{T}$ is a collection of open sets such that for each open set $U \in \mathscr{T}$ of X and for each $x \in U$, there is an element $V \in \mathscr{C}$ such that $x \in V \subseteq U$. Then $\mathscr{C}$ is a basis for $\mathscr{T}$.*

Proof. Let $X, \mathscr{T}$ be a topological space and let $\mathscr{C}$ be a collection of open sets as specified above. We have two tasks: to show $\mathscr{C}$ is a basis, and to show $\mathscr{C}$ generates $\mathscr{T}$.

To show $\mathscr{C}$ is a basis, let $x \in X$ be given. As X is open, by definition of $\mathscr{C}$ there exists a $V \in \mathscr{C}$ such that $x \in V$. Let $V_1, V_2 \in \mathscr{C}$. As V_1, V_2 are open in $\mathscr{T}$, $U = V_1 \cap V_2$ is open. Again by the definition of $\mathscr{C}$, for all $x \in U$ there exists $V \in \mathscr{C}$ such that $x \in V \subseteq U = V_1 \cap V_2$. Thus $\mathscr{C}$ satisfies Definition 5.1.

Let $\mathscr{T}'$ be the topology generated by $\mathscr{C}$. We will now show that $\mathscr{T}' = \mathscr{T}$. Let $U \in \mathscr{T}'$. By Lemma 5.6, $U = \bigcup_{V \in \mathscr{C}'} V$ for some subcollection $\mathscr{C}' \subseteq \mathscr{C}$. As each V is open in $\mathscr{T}$, the arbitrary union U is in $\mathscr{T}$. Thus $\mathscr{T}' \subseteq \mathscr{T}$. Let $U \in \mathscr{T}$. By definition of $\mathscr{C}$, for each $x \in U$ there exists $V \in \mathscr{C}$ such that $x \in V \subseteq U$. Thus $U \in \mathscr{T}'$ by Definition 5.2. Hence, $\mathscr{T} \subseteq \mathscr{T}'$ and we conclude that equality holds. □

With regards to comparable topologies, bases behave as one would hope.

Lemma 5.8. *Let $\mathscr{B}$ and $\mathscr{B}'$ be bases for topologies $\mathscr{T}$ and $\mathscr{T}'$ respectively on some set X. Then the following are equivalent:*

i) *$\mathscr{T}'$ is finer than $\mathscr{T}$.*
ii) *For each $x \in X$ and each basis element $B \in \mathscr{B}$ containing x, there is a basis element $B' \in \mathscr{B}'$ such that $x \in B' \subseteq B$.*

Proof. [i) $\Rightarrow$ ii)] Suppose $\mathscr{T}'$ is finer that $\mathscr{T}$. Let $x \in X$ and $B \in \mathscr{B}$ with $x \in B$ be given. As B is open in $\mathscr{T}$ and $\mathscr{T}'$ is finer than $\mathscr{T}$, we have that B is open in $\mathscr{T}'$. By Definition 5.2, there is a basis element $B' \in \mathscr{B}'$ such that $x \in B' \subseteq B$. Hence the ii) holds.

[ii) $\Rightarrow$ i)] Suppose that $\mathscr{B}$ and $\mathscr{B}'$ have the property that for all $x \in X$ and each basis element $B \in \mathscr{B}$ containing x, there is a basis element $B' \in \mathscr{B}'$ such that $x \in B' \subseteq B$. Let $U \in \mathscr{T}$ and let $x \in U$. We will show that $U \in \mathscr{T}'$ via Definition 5.2. As $\mathscr{B}$ generates $\mathscr{T}$, there is a $B \in \mathscr{B}$ such that $x \in B \subseteq U$. By supposition, there is a basis element $B' \in \mathscr{B}'$ such that $x \in B' \subseteq B$. Then $x \in B' \subseteq U$ and U is open in $\mathscr{T}'$. We then conclude that $\mathscr{T} \subseteq \mathscr{T}'$ as required. □

As Lemma 5.8 gives a comparison of topologies via bases, it also provides a means by which to conclude if two bases generate the same topology. That is, if statement ii) holds for both the basis $\mathscr{B}$ and $\mathscr{B}'$, then $\mathscr{T} = \mathscr{T}'$.

We will now define three topologies on $\mathbb{R}$ that will appear throughout the remainder of the course.

Definition 5.9. Consider $\mathbb{R}$ with the strict total order $<$. Let $\mathscr{B} = \{(a, b) \subset \mathbb{R}\}$. The topology generated by $\mathscr{B}$ is the *standard topology* on $\mathbb{R}$.

The standard topology is also sometimes referred to as the usual topology and denoted $\mathbb{R}_{\mathscr{U}}$. It is safe to assume that if $\mathbb{R}$ is being considered as a topological space and no explicit mention of a particular topology has been made then $\mathbb{R}$ has been given the standard topology.

Definition 5.10. Let $\mathscr{B}' = \{[a, b) \subset \mathbb{R}\}$. The topology generated by $\mathscr{B}'$ is the *lower limit topology* on $\mathbb{R}$. We denote $\mathbb{R}$ together with the lower limit topology by the symbol $\mathbb{R}_l$.

Definition 5.11. Let $K = \{\frac{1}{n} : n \in \mathbb{Z}_+\}$ and let

$$\mathscr{B}'' = \{(a, b) - K : (a, b) \subset \mathbb{R}\} \cup \{(a, b) \subset \mathbb{R}\}.$$

The topology generated by $\mathscr{B}''$ is called the *K-topology* on $\mathbb{R}$. When $\mathbb{R}$ is given the K topology we denote this by $\mathbb{R}_K$.

The comparison of $\mathbb{R}_l$, $\mathbb{R}_K$, and the standard topology should be straightforward with the aid of Lemma 5.8.

By Lemma 5.6, we can think of the topology generated by a basis as the collection of arbitrary unions of basis elements. This agrees well with the axioms of topology, one of them being closure under arbitrary unions. This raises the question of what constructions can be made if we instead take finite intersections of sets in a collection.

Definition 5.12. Let X be a set and let $\mathcal{S}$ be a collection of subsets of X such that the union over all elements of $\mathcal{S}$ is X. We call $\mathcal{S}$ a *subbasis* for a topology on X. The *topology generated by the subbasis* $\mathcal{S}$ is defined to be the collection

$$\mathscr{T} = \left\{ U \subseteq X : U = \bigcup_{\alpha \in J} \bigcap_{S_i \in \mathcal{S}_\alpha} S_i \right\}$$

where J is some index and for all $\alpha \in J$, $\mathcal{S}_\alpha$ is a finite subcollection of $\mathcal{S}$.

Like for Definition 5.2, we need to show that the topology generated by a subbasis is actually a topology. The proof should look more mature now that we have more experience and a larger toolbox.

Lemma 5.13. *Let X be a set and let $\mathcal{S}$ be a subbasis. The collection*

$$\mathscr{T} = \left\{ U \subseteq X : U = \bigcup_{\alpha \in J} \bigcap_{S_i \in \mathcal{S}_\alpha} S_i \right\}$$

where J is some index and for all $\alpha \in J$, $\mathcal{S}_\alpha$ is a finite subcollection of $\mathcal{S}$ is a topology on X.

Proof. Let X be a set and $\mathcal{S}$ a subbasis. Define

$$\mathscr{B} = \left\{ U \subseteq X : U = \bigcap_{S_i \in \mathcal{S}'} S_i \right\}.$$

We claim that $\mathscr{B}$ is a basis. If $\mathscr{B}$ is a basis, then $\mathscr{T}$ being the union of elements of $\mathscr{B}$ is the topology generated by $\mathscr{B}$, in particular, $\mathscr{T}$ is a topology on X.

Let $x \in X$ be given. As $X = \bigcup_{S \in \mathcal{S}} S$ then for some $S \in \mathcal{S}$, $x \in S$. As S is trivially an intersection with itself, then $S \in \mathscr{B}$. Hence $\mathscr{B}$ satisfies the first axiom of being a basis. Let $B_1 = S_1 \cap S_2 \cap \cdots \cap S_n$ and $B_2 = S'_1 \cap S'_2 \cap \cdots \cap S'_m$ and suppose that $x \in B_1 \cap B_2$. We see $B_1 \cap B_2 = (S_1 \cap S_2 \cap \cdots \cap S_n) \cap (S'_1 \cap S'_2 \cap \cdots \cap S'_m) \in \mathscr{B}$. Letting $B_3 = B_1 \cap B_2$, we see that $\mathscr{B}$ satisfies the second axiom and is thus a basis for the topology $\mathscr{T}$. □

6. Order, Product, and Subspace Topologies

6.1. **Introduction.** In this section we will look at ways of constructing new topologies from the ideas explored in Sections 1 and 3, using set theory and order relations.

6.2. **The Order Topology.** Recall for a set X with a strict total order $<$ the definitions of open, closed, and half-open intervals:

Definition 3.4. *Let $<$ be a strict total order on a set X. If $a < b$ we denote the* ***open interval*** *as*

$$(a, b) = \{x \in X : a < x < b\},$$

the ***closed interval*** *as*

$$[a, b] = \{x \in X : a \leq x \leq b\},$$

and the ***half-open intervals*** *as*

$$[a, b) = \{x \in X : a \leq x < b\}$$
$$(a, b] = \{x \in X : a < x \leq b\}$$

The term open interval suggests a connection to open sets for some particular topology. We are already familiar with one example of such a topology, which we will build now.

Definition 6.1. Let X be a set with at least two elements with a strict total order. Let $\mathscr{B}$ be the collection of all subsets of X of the form:

i) All open intervals of (a, b) of X.
ii) All intervals of the form $[a_0, b)$ where a_0 is the smallest element of X, should it exist.
iii) All intervals of the form $(a, b_0]$ where b_0 is the largest element of X, should it exist.

The collection $\mathscr{B}$ is a basis for a topology on X called the *order topology*.

If X has no smallest element then $\mathscr{B}$ has no elements of type ii). Likewise, if X has no largest element then $\mathscr{B}$ has no elements of type iii). Because these notes have dealt (quite explicitly) with total orders, if a set has a largest (smallest) element then it is unique.

The reader is encouraged to think of the effect that allowing the order to be a strict partial order will have on the basis (and hence the generated topology). Further in the case that X has exactly two elements, can the reader describe the elements of types ii) and iii)?

We have still to verify the claim that $\mathscr{B}$ in Definition 6.1 is a basis.

Claim 6.2. *Let X and $\mathscr{B}$ be as in Definition 6.1. Then $\mathscr{B}$ is a basis for a topology on X.*

Proof. We will assume that X has both a smallest element a_0 and a largest element b_0. The cases in which X has only one or none of the latter two such elements will follow immediately.

We have two axioms to check from Definition 5.1.

(1) Let $x \in X$ be given. If x is not the smallest element, then $x \in (a_0, b_0]$. If x is not the largest element, then $x \in [a_0, b_0)$.
(2) Let $B_1, B_2 \in \mathscr{B}$. If $B_1 \cap B_2 \neq \emptyset$, we will show that $B_1 \cap B_2 \in \mathscr{B}$. Suppose that $B_1 = (a_1, b_1), B_2 = (a_2, b_2)$ are both elements of type i). Without loss of generality, assume $a_1 \leq a_2$. If $b_1 \leq a_2$, then $B_1 \cap B_2 = \emptyset$. Otherwise $B_1 \cap B_2 = (a_2, \min\{b_1, b_2\}) \in \mathscr{B}$. The remaining intersections follow similar reasoning and are tabulated below:

$$(a_1, b_1) \cap (a_2, b_0] = \begin{cases} \emptyset \text{ if } b_1 \leq a_2 \\ (\max\{a_1, a_2\}, b_1) \text{ if } a_2 < b_1 \end{cases}$$

$$[a_0, b_1) \cap (a_2, b_2) = \begin{cases} \emptyset \text{ if } b_1 \leq a_2 \\ (a_2, \min\{b_1, b_2\}) \text{ if } a_2 < b_1 \end{cases}$$

$$[a_0, b_1) \cap (a_2, b_0] = \begin{cases} \emptyset \text{ if } b_1 \leq a_2 \\ (a_2, b_1) \text{ if } a_2 < b_1 \end{cases}$$

Each of these intersections, when non-empty, are in $\mathscr{B}$. Hence, $\mathscr{B}$ satisfies both axioms in Definition 5.1 and is thus a basis for a topology on X.

□

Example 6.3. The standard topology on $\mathbb{R}$ is the order topology derived from the usual strict total order $<$ on $\mathbb{R}$.

Example 6.4. By the Well-ordering Theorem, found in Section 10 of Munkres and proven (to much commotion) in 1904 by Zermelo, there is an uncountable set with a strict total order that is well ordering. Denote by ω_1 the uncountable well-ordered set such that if X is uncountable then there exists an injection $f : \omega_1 \to X$. In this way, we can consider ω_1 as the smallest uncountable set. Due to the well-ordering, ω_1 has a smallest element which we call 0.

Let X be the Cartesian product $X = \omega_1 \times [0, 1)$ where $[0, 1)$ is the half-open interval in $\mathbb{R}$. The space X with the order topology given by the dictionary order on the Cartesian product is called the *closed long ray*.

Example 6.5. The positive integers $\mathbb{Z}_+$ are an ordered set with a smallest element 1. The set $[1, 2) = \{1\}$ is a basic open set in the order topology as is $(n-1, n+1) = \{n\}$. By Lemma 5.8, the order topology is finer than the discrete topology. As the discrete topology is the finest topology, the two topologies must be equal.

Example 6.6. The set $X = \{1, 2\} \times \mathbb{Z}_+$ with the dictionary order is another set with a smallest element 1×1. When considering Cartesian products with order relations we typically write elements as $a \times b$ to not cause confusion with open intervals (x, y). The order topology on X is not the discrete topology, as the singleton $\{2 \times 1\}$ is not open (any open interval containing 2×1 must contain elements from $\{1\} \times \mathbb{Z}_+$).

Example 6.7. Let $Y = \mathbb{Z}_+ \times \{1, 2\}$ with the dictionary order. Again, Y has a smallest element, 1×1. In contrast to Example 6.6, the order topology on Y is the discrete topology: one need only consider the basic open sets $(n - 1 \times 2, n \times 2) = \{n \times 1\}, (n \times 1, n + 1 \times 1) = \{n \times 2\}$, and $[1 \times 1, 1 \times 2) = \{1 \times 1\}$.

Definition 6.8. If X is a set with a strict total order, we can associate to each element $a \in X$ four subsets: The *open rays*

$$(a, +\infty) = \{x \in X : x > a\}$$
$$(-\infty, a) = \{x \in X : x < a\}$$

and the *closed rays*

$$[a, +\infty) = \{x \in X : x \geq a\}$$
$$(-\infty, a] = \{x \in X : x \leq a\}$$

Remark 6.9. Note if X has a largest element b_0, then open ray (a, ∞) and the interval $(a, b_0]$ are the same subset of X. Likewise for the other rays and for the presence of a smallest element.

Example 6.10. Let X be a set with a strict total order. The set $\mathcal{S}$ of all open rays forms a subbasis for the order topology on X.

Example 6.11. Let X be a set with a strict total order. The set $\mathscr{B} = \{(a, +\infty) : a \in X\}$ is a basis for the *open half-line topology*. It is left to the reader to show that $\mathscr{B}$ is a basis and that the topology generated by $\mathscr{B}$ is strictly coarser than the order topology on X.

6.3. **The Product Topology.** In this section we will look at a way to define a topology on a Cartesian product of two topological spaces. First, a note on notation. For the first time since defining a topological space, we wish to discuss two spaces simultaneously. There are a few options on how to proceed. If the topologies associated to two spaces X and Y need never be invoked by name, then simply the remark that both X and Y are topological spaces will suffice. But if the topologies themselves need to be referenced, a nice convention is to use the underlying point-set in the name. Then $\mathscr{T}_X$ is likely the topology on X and $\mathscr{T}_Y$ is as likely to be the topology on Y. Some authors prefer to use $\mathscr{F}$ for their second topology, and if bases are needed use both $\mathscr{B}$ and $\mathscr{C}$. We will prefer to again name our bases after the associated point-set: $\mathscr{B}_X, \mathscr{B}_Y$.

Definition 6.12. Let $X, \mathscr{T}_X$ and $Y, \mathscr{T}_Y$ be topological spaces and let $\mathscr{B} = \{U \times V \in \mathscr{T}_X \times \mathscr{T}_Y\}$. Then $\mathscr{B}$ is a basis and we define the *product topology* to be the topology on $X \times Y$ generated by $\mathscr{B}$.

As each $U \in \mathscr{T}_X$ is a subset of X and each $V \in \mathscr{T}_Y$ is a subset of Y, each set of the form $U \times V$ is a subset of $X \times Y$. We leave it to the reader to verify that $\mathscr{B}$ is a basis. We note that $\mathscr{B}$ is not in general a topology itself, as the union $(U_1 \times V_1) \cup (U_2 \times V_2)$ need not be a product of two subsets of X and Y resp.

Definition 6.12 works well if both $\mathscr{T}_X$ and $\mathscr{T}_Y$ have been defined explicitly. If, as is often the case, we are instead given bases $\mathscr{B}_X$ and $\mathscr{B}_Y$, then we can construct another basis for the product topology:

Theorem 6.13. *Let $\mathscr{B}_X$ be a basis for the topology on X and $\mathscr{B}_Y$ be a basis for the topology on Y. The collection*

$$\mathscr{D} = \{B \times B' \in \mathscr{B}_X \times \mathscr{B}_Y\}$$

is a basis for the product topology on $X \times Y$

Proof. As we are given a description of the product topology in Definition 6.12, we can use Lemma 5.7 to show that $\mathscr{D}$, a collection of open sets, is a basis for $X \times Y$. Let $\mathscr{T}$ be the product topology on $X \times Y$ and denote its basis as $\mathscr{B} = \{U \times V \in \mathscr{T}_X \times \mathscr{T}_Y\}$. Let $W \in \mathscr{T}$ be an open subset of $X \times Y$ and let $x_0 \times x_1 \in W$ be given. As $\mathscr{B}$ is a basis of $\mathscr{T}$ there exists a basis element $U \times V \subseteq W$ such that $x_0 \times x_1 \in U \times V$. As $\mathscr{B}_X$ is a basis for $\mathscr{T}_X$ there exists some $B \in \mathscr{B}_X$ such that $x_0 \in B \subseteq U$. Similarly there exists some $B' \in \mathscr{B}_Y$ such that $x_1 \in B' \subseteq V$. Then $x_0 \times x_1 \in B \times B' \subseteq U \times V \subseteq W$. As both W and $x_0 \times x_1 \in W$ are arbitrary, this holds for all such W. By Lemma 5.7, $\mathscr{D}$ is a basis for $\mathscr{T}$, the product topology on $X \times Y$. □

Example 6.14. The product topology on $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ where each $\mathbb{R}$ is given the standard topology is referred to as the *standard topology on* $\mathbb{R}^2$. By Theorem 6.13, the standard topology has a basis consisting of all products of open intervals $(a_0, b_0) \times (a_1, b_1)$.

Example 6.15. The product topology on two spaces with order topologies is not the order topology given by the dictionary order. Consider the space $X = \{1, 2\} \times \mathbb{Z}_+$ from Example 6.6. Both $\{1, 2\}$ and $\mathbb{Z}_+$ with the order topologies are the discrete topology on each respective set. The product topology is then again the discrete topology on X. As we saw in Example 6.6, the order topology on X is not the discrete topology.

We can also express the product topology through a subbasis. For this we will need two important functions

Definition 6.16. Let $\pi_1 : X \times Y \to X$ be the function given by the rule $\pi_1(x, y) = x$ and let $\pi_2 : X \times Y \to Y$ be given by the rule $\pi_2(x, y) = y$. The functions π_1, π_2 are the *projections* of $X \times Y$ onto the first and second factors.

If U is open in X, then the preimage $\pi_1^{-1}(U) = U \times Y$ is open in $X \times Y$. Similarly for open V, $\pi_2^{-1}(V) = X \times V$ is open in $X \times Y$.

Theorem 6.17. *The collection*

$$\mathcal{S} = \{\pi_1^{-1}(U) : U \in \mathscr{T}_X\} \cup \{\pi_2^{-1}(V) : V \in \mathscr{T}_Y\}$$

is a subbasis for the product topology on $X \times Y$.

Proof. Let $\mathscr{T}$ denote the product topology on $X \times Y$. As $\pi_1^{-1}(U) = U \times Y$ and $\pi_2^{-1}(V) = X \times V$, then $X \times Y = \bigcup_{S \in \mathcal{S}} S$. Thus $\mathcal{S}$ is a subbasis. Let $\mathscr{B}_{\mathcal{S}}$ be the collection of all finite intersections of elements of $\mathcal{S}$. By the proof of Lemma 5.13, $\mathscr{B}_{\mathcal{S}}$ is a basis for the topology generated by $\mathcal{S}$. It suffices to show that $\mathscr{B}_{\mathcal{S}}$ is a basis for $\mathscr{T}$.

As each element of $\mathcal{S}$ is open in $\mathscr{T}$, each element of $\mathscr{B}_{\mathcal{S}}$, being a finite intersection of open sets, is again open in $\mathscr{T}$. Let $W \in \mathscr{T}$ and $x \in W$ be given (the fact that x is an ordered pair is irrelevant). As $\mathscr{B} = \{U \times Y \in \mathscr{T}_X \times \mathscr{T}_Y\}$ is a basis for $\mathscr{T}$, there exists a basis element $U \times V \in \mathscr{B}$ such that $x \in U \times V \subseteq W$. As $U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V)$, we see that $U \times V \in \mathscr{B}_{\mathcal{S}}$. Thus by Lemma 5.7, $\mathscr{B}_{\mathcal{S}}$ is a basis for $\mathscr{T}$. □

6.4. **The Subspace Topology.** The final topologies we will construct in the section are on subsets of a given topological space.

Definition 6.18. Let $X, \mathscr{T}_X$ be a topological space. If Y is a subset of X, the collection $\mathscr{T}_Y = \{Y \cap U : U \in \mathscr{T}_X\}$ is a topology on Y called the *subspace topology*. When Y is considered with this topology, we call Y a *subspace* of X.

It is straightforward to verify that $\mathscr{T}_Y$ is a topology.

The subspace topology is given by a predictable basis and subbasis:

Lemma 6.19. *Let $\mathscr{B}_X$ be a basis for the topology $\mathscr{T}_X$ on X and let Y be a subspace of X. The collection $\mathscr{B}_Y = \{Y \cap B : B \in \mathscr{B}_X\}$ is a basis for the subspace topology $\mathscr{T}_Y$.*

Proof. As $B \in \mathscr{B}_X$ is a basic open set in X, $Y \cap B$ is open in the subspace Y. Thus $\mathscr{B}_Y$ is a collection of open sets in Y. Let $V \in \mathscr{T}_Y$ be an open subset of Y and let $y \in V$ be given. By Definition 6.18, there exists some $U \in \mathscr{T}_X$ such that $V = Y \cap U$. As $\mathscr{B}_X$ is a basis for $\mathscr{T}_X$, there exists some $B \in \mathscr{B}_X$ such that $y \in B \subseteq U$. Then $y \in Y \cap B \subseteq Y \cap U = V$. It follows from Lemma 5.7 that $\mathscr{B}_Y$ is a basis for $\mathscr{T}_Y$, the subspace topology on Y. □

Lemma 6.20. *Let $\mathcal{S}_X$ be a subbasis for a topology on X. Then the collection $\mathcal{S}_Y = \{Y \cap S : S \in \mathcal{S}_X\}$ is a subbasis for the subspace topology on Y.*

Proof. The topology on X generated by $\mathcal{S}_X$ is generated by the basis $\mathscr{B}_X$ consisting of finite intersections of elements of $\mathcal{S}_X$. Let $\mathscr{B}_Y$ be the collection of all finite intersections of elements of $\mathcal{S}_Y$. Each $B \in \mathscr{B}_Y$ is of the form $B = \bigcap_{i=1}^n (Y \cap S_i) = Y \cap (\bigcap_{i=1}^n S_i)$. As $\bigcap_{i=1}^n S_i \in \mathscr{B}_X$, B is open in Y.

Let U be open in the subspace topology on Y and let $y \in U$ be given. By definition there exists some open set V of X such that $U = Y \cap V$. As such, $y \in V$ and there must exist some $B = \bigcap_{i=1}^n S_i \in \mathscr{B}_X$ such that $y \in B \subset V$. We then have $y \in Y \cap B = Y \cap \bigcap_{i=1}^n S_i \subseteq U$. As $Y \cap \bigcap_{i=1}^n S_i \in \mathscr{B}_Y$, by Lemma 5.7, $\mathscr{B}_Y$ is a basis for the subspace topology on Y. Hence $\mathcal{S}_Y$ is a subbasis for the subspace topology on Y. □

The open sets of Y do not look like open sets in X in general.

Example 6.21. Consider $\mathbb{R}$ with the usual topology and let $Y = [0, 2]$. Then $[0, 2] \cap (-1, 1) = [0, 1)$ is open in the subspace Y, but is not open in $\mathbb{R}$.

There is an important instance in which the open sets of Y coincide with open sets of X.

Lemma 6.22. *If U is open in Y and Y is open in X, then U is open in X.*

Proof. Suppose U is open in Y. Then by Definition 6.18, there is some $V \in \mathscr{T}_X$ such that $U = Y \cap V$. If Y is open in X then $U = Y \cap V$ is the intersection of open sets in X and hence is itself open in X. □

Subspace topologies behave as one might expect in product topologies.

Theorem 6.23. *Let A be a subspace of X and let C be a subspace of Y. The product topology on $A \times C$ is the same topology that $A \times C$ inherits as a subspace of $X \times Y$.*

Proof. Fix bases $\mathscr{B}_X$ and $\mathscr{B}_Y$ of $\mathscr{T}_X$ and $\mathscr{T}_Y$ respectively. The bases for the subspaces A and C are given by $\mathscr{B}_A = \{A \cap B : B \in \mathscr{B}_X\}$ and $\mathscr{B}_C = \{C \cap B' : B' \in \mathscr{B}_Y\}$. Let

$\mathscr{T}_{\text{prod}}$ denote the product topology on $A \times C$. By Theorem 6.13, $\mathscr{D} = \{(A \cap B) \times (C \cap B') : B \in \mathscr{B}_A \text{ and } B' \in \mathscr{B}_C\}$ is a basis for $\mathscr{T}_{\text{prod}}$.

Let $\mathscr{T}_{A\times C}$ denote the subspace topology inherited by $A \times C$ from the topology on $X \times Y$. By Lemma 6.19 and Theorem 6.13, $\mathscr{B}_{A\times C} = \{(A \times C) \cap (B \times B') : B \in \mathscr{B}_X \text{ and } B' \in \mathscr{B}_Y\}$ is a basis for $\mathscr{T}_{A\times C}$.

As $(A \times C) \cap (B \times B') = (A \cap B) \times (C \cap B')$, the collections $\mathscr{D}$ and $\mathscr{B}_{A\times C}$ are equal as sets. Hence they generate the same topology. □

The interaction of order topologies and subspace topologies is not so obvious. If X is a set with a strict total order $<_X$, we saw in Section 3.2 that any subset $Y \subseteq X$ inherits a strict total order $<_Y$. It is not necessarily the case that the order topology on Y given by the strict total order $<_Y$ is the same as the subspace topology on Y inherited from the order topology on X.

Example 6.24. Consider the subset $Y = [0, 1) \cup \{2\}$ of $\mathbb{R}$. In the subspace topology $\mathscr{T}_Y$, the singleton $\{2\}$ is open as it equals the intersection $Y \cap (\frac{3}{2}, 3)$. In the order topology on Y, every open set containing 2 must have as a subset some open interval of the form $(a, 2]$. No such set is equal to the singleton $\{2\}$. *N.B.* there is no interval of the form $(1, 2]$ in Y as $1 \notin Y$.

Despite the seeming oddness of the order topology, such consternations as in Example 6.24 do not occur for open intervals or rays.

Definition 6.25. Given a set X with a strict total order, a subset Y of X is called *convex* in X if every pair $a, b \in Y$ such that $a < b$ the open interval (a, b) of X is a subset of Y.

Theorem 6.26. *Let X be a set with a strict total order $<_X$ and let $\mathscr{T}_X$ be the induced order topology on X. Let Y be a subset of X that is convex. Then the order topology on Y is the same as the topology on Y inherited as a subspace of X.*

Proof. Let $\mathscr{T}_<$ denote the order topology on Y and $\mathscr{T}_Y$ denote the subspace topology on Y. Let $a \in X$ be given. We have the following observations:

$$(a, +\infty) \cap Y = \begin{cases} \{x \in Y : x > a\} \text{ if } a \in Y \\ \emptyset \text{ if } \forall x \in Y\ a \geq x \\ Y \text{ if } \forall x \in Y\ a < x \end{cases} \tag{6.1}$$

$$(-\infty, a) \cap Y = \begin{cases} \{x \in Y : x < a\} \text{ if } a \in Y \\ \emptyset \text{ if } \forall x \in Y\ a \leq x \\ Y \text{ if } \forall x \in Y\ a > x \end{cases} \tag{6.2}$$

Where the second and third equality in (6.1) and (6.2) follow by convexity of Y.

By Lemma 6.20 and Example 6.10, the sets $(a, +\infty) \cap Y$ and $(-\infty, a) \cap Y$ form a subbasis of $\mathscr{T}_Y$. As each of the sets in (6.1) and (6.2) above are open in the order topology, the order topology is finer than the subspace topology.

For $a \in Y$, let $U_a = \{x \in Y : x > a\}$ be an open ray in Y. As $a \in Y \subseteq X$, it follows that $a \in X$ and hence $U_a = (a, +\infty) \cap Y$. Likewise every open ray of the form $\{x \in Y : x < a\}$ can be expressed as an intersection $(-\infty, a) \cap Y$. As the open rays of Y form a subbasis for $\mathscr{T}_<$ and each of these intersections are by definition open in $\mathscr{T}_Y$, then the subspace topology is finer than the order topology.

To conclude, we have shown that $\mathscr{T}_Y \subseteq \mathscr{T}_<$ and $\mathscr{T}_< \subseteq \mathscr{T}_Y$ and hence both topologies on Y are the same. □

In general, the subspace topology will find more use than the order topology on a subset. Whence, unless otherwise stated, the topology on a subset is assumed to be the subspace topology.

7. Closed Sets and Limit Points

7.1. **Introduction.** Thus far, our study of topological spaces has focused entirely on open sets: topologies are defined as a collection of open sets, we build topologies with a 'simple' collection of open sets, we use open sets to define new topologies, etc. In this section, we will look at how other subsets interact with the notion topology. We will define closed sets, limit points, and the first two separation axioms.

7.2. Closed Sets.

Definition 7.1. A subset A of a topological space $X, \mathscr{T}$ is *closed* if the set $X - A$ is open.

So, our first characterization of closed sets is in terms of open sets. It should be seen immediately that had we discovered a notion of closedness first, we could have defined open sets in terms of those which are closed. We will expand on this after a collection of examples.

Example 7.2. The subset $[a, b]$ of $\mathbb{R}$ (with the standard topology) is closed as the complement

$$\mathbb{R} - [a, b] = (\infty, a) \cup (b, \infty)$$

is open. Thus the name closed interval. Likewise $[a, \infty)$ is closed as

$$\mathbb{R} - [a, \infty) = (-\infty, a)$$

is open.

Example 7.3. The horizontal line $L = \{(x, 1) \in \mathbb{R}^2\}$ is closed with respect to the standard topology on $\mathbb{R}^2$ as

$$\mathbb{R}^2 - L = [\mathbb{R} \times (-\infty, 1)] \cup [\mathbb{R} \times (1, \infty)]$$

is the open.

Example 7.4. In the discrete topology on a set X, every set is closed. If A is a subset of X, then $X - A$ is a subset of X and is then open.

Example 7.5. Let $Y = [0, 1] \cup (2, 3)$ have the subspace topology induced as a subset of $\mathbb{R}$ with the standard topology. The subset $[0, 1]$ is closed in Y as $Y - [0, 1] = (2, 3) = Y \cap (2, 3)$ is open. Likewise the subset $(2, 3)$ is closed as $Y - (2, 3) = [0, 1] = Y \cap (-1, \frac{3}{2})$ is open. The sets $[0, 1]$ and $(2, 3)$ are both open and closed in the subspace topology on Y. Some authors will refer to such sets with the portmanteau "clopen."

Example 7.6. For any discrete topology, one-point sets are open. in fact, we recall that the collection of all singletons forms a basis for the discrete topology. Singletons in the standard topology on $\mathbb{R}$, while certainly not open, are in fact closed as $\mathbb{R} - \{x\} = (-\infty, x) \cup (x, \infty)$ for all $x \in \mathbb{R}$. Not all topologies have closed singletons, and it turns out there are nice consequences to such a property.

As it so happens, the collection of all closed subsets of a space X satisfies certain properties resembling the axioms of topologies (Definition 4.1). In fact, we could have developed all theory up to this point using the following (compare to Theorem 17.1 in Munkres):

Theorem 7.7. *Let $X, \mathscr{T}$ be a topological space and let $\mathscr{F}$ be the collection all closed subsets of X. Then $\mathscr{F}$ satisfies:*

i) $\emptyset \in \mathscr{F}$ *and* $X \in \mathscr{F}$.
ii) *If $\mathscr{A} \subseteq \mathscr{F}$ is any subcollection of $\mathscr{F}$, then $\bigcap_{A\in\mathscr{A}} A \in \mathscr{F}$.*
iii) *If $\{A_1, A_2, \ldots, A_n\} \subseteq \mathscr{F}$ is a finite subcollection of $\mathscr{F}$ then $\bigcup_{i=1}^{n} A_i \in \mathscr{F}$.*

Another way of stating this theorem is that the empty-set and X are closed, arbitrary intersections of closed sets are closed, and finite unions of closed sets are closed.

Proof. Let the topological space $X, \mathscr{T}$ be given and let $\mathscr{F}$ be the collection all closed subsets of X. The definition of $\mathscr{F}$ is such that $A \in \mathscr{F} \iff X - A \in \mathscr{T}$.

i) $X - \emptyset = X \in \mathscr{T} \Rightarrow \emptyset \in \mathscr{F}$ and $X - X = \emptyset \in \mathscr{T} \Rightarrow X \in \mathscr{F}$.
ii) Let $\mathscr{A} \subseteq \mathscr{F}$ be a subcollection of $\mathscr{F}$. Then by DeMorgan's Laws $X - \bigcap_{A\in\mathscr{A}} A = \bigcup_{A\in\mathscr{A}}(X - A) \in \mathscr{T}$ as each $X - A \in \mathscr{T}$. Hence $\bigcap_{A\in\mathscr{A}} A \in \mathscr{F}$.
iii) Let $\{A_1, A_2, \ldots, A_n\} \subseteq \mathscr{F}$ be a finite subcollection of $\mathscr{F}$. Then $X - \bigcup_{i=1}^{n} A_i = \bigcap_{i=1}^{n}(X - A) \in \mathscr{T}$ as each $X - A \in \mathscr{T}$. Hence $\bigcup_{i=1}^{n} A_i \in \mathscr{F}$.

□

The closed sets of a subspace are convenient enough to work with, though we should be careful with verbiage. We say $A \subseteq Y \subseteq X$ is **closed in** Y if we wish to view it as a closed subset of the subspace Y, and **closed in** X if we need to specify it as a closed subset of the space X. Example 7.5 shows this distinction to be important.

Theorem 7.8. *Let $Y, \mathscr{T}_Y$ be a subspace of $X, \mathscr{T}_X$. Then a subset $A \subseteq Y$ is closed in Y if and only if $A = Y \cap C$ for some set C closed in X.*

Proof. ($\Rightarrow$) Suppose A is closed in Y. Then $Y - A$ is open in the subspace topology. By definition of the subspace topology there exists some $U \in \mathscr{T}_X$ such that $Y \cap U = Y - A$. Let $C = X - U$. Clearly C is closed and $Y \cap C = Y \cap (X - U) = Y - U = Y \cap A = A$. Whence, A is the intersection of Y with some set C that is closed in X.

($\Leftarrow$) Let C be closed in X and let $A = Y \cap C$. Then $Y - A = Y - C = Y \cap (X - C)$. As C is closed in X, $X - C \in \mathscr{T}_X$. Hence $Y - A = Y \cap (X - C) \in \mathscr{T}_Y$. Thus A is closed in Y. □

Mirroring Lemma 6.22, we have the following result about closed set in subspaces:

Lemma 7.9. *If A is closed in Y and Y is closed in X, then A is closed in X.*

Proof. By Theorem 7.8 $A = Y \cap C$ for some set C closed in X. By property iii) of Theorem 7.2 $Y \cap C$ is closed in X, therefore A is closed in X. □

7.3. **Closures, Interiors, Exteriors, and Boundaries.** Given a subset A of a topological space X, we create certain open and closed sets relating to A.

Definition 7.10. Let X be a topological space and A a subset of X.

(1) Let $\mathcal{C}$ be the collection of all closed sets C such that $A \subseteq C$. The *closure* of A, denoted Cl A or $\bar{A}$, is the set $\bar{A} = \bigcap_{C\in\mathcal{C}} C$. That is, the closure of A is the intersection of all closed sets containing A.

(2) Let $\mathscr{U}$ be the collection of all open sets U such that $U \subseteq A$. The *interior* of A, denoted Int A or $\mathring{A}$, is the set $\mathring{A} = \bigcup_{U \in \mathscr{U}} U$. That is, the interior of A is the union of all open sets contained in A.
(3) The *boundary* of A, denoted Bd A or ∂A, is the set $\partial A = \bar{A} \cap \overline{(X - A)}$.
(4) The *exterior* of A, denoted Ext A, is the set Ext A = Int $(X - A)$. The exterior is the interior of the complement.

The associations and interactions of these sets is explored in Munkres Exercise 17.19. When a subspace Y is involved, we may want to take the closure of a set $A \subseteq Y$. The notation $\bar{A}$ is reserved for the closure of A in X. We can easily represent the closure of A in Y in terms of $\bar{A}$:

Theorem 7.11. *Let Y be a subspace of X and let A be a subset of Y. The closure of A in Y equals $\bar{A} \cap Y$.*

Proof. Let B be the closure of A in Y.

($B \subseteq \bar{A} \cap Y$) The set $\bar{A}$ is by definition closed in X. By Theorem 7.8, $\bar{A} \cap Y$ is closed in Y. As $A \subseteq \bar{A} \cap Y \subseteq Y$ and B is defined as the intersection of all closed sets (in Y) that contain A, we see that $B \subseteq \bar{A} \cap Y$.

($\bar{A} \cap Y \subseteq B$) The set B is closed in Y. By Theorem 7.8, there exists some set $C \subseteq X$ that is closed in X such that $B = C \cap Y$. Since $A \subseteq B$, $A \subseteq C$. As $\bar{A}$ is the intersection of all closed sets (in X) containing A, we have $\bar{A} \subseteq C$. Thus $\bar{A} \cap Y \subseteq C \cap Y = B$.

We have now shown that $B = \bar{A} \cap Y$. As B was defined as the closure of A in Y, we have completed the proof. □

Our definition of closure (and interior for that matter) is non-constructive. In general, one expects there to be A LOT of closed sets containing some subset A. The intersection of this arbitrary collection of closed sets may be impractical to compute. There is another way to describe the closure in terms of open sets, and by consequence in terms of basic open sets. It is then hopefully the case that the "fewer" basic open sets one needs to deal with makes the computation of $\bar{A}$ more manageable.

Theorem 7.12. *Let A be a subset of the topological space $X, \mathscr{T}$ and suppose $\mathscr{B}$ is a basis for $\mathscr{T}$.*

(1) *Then $x \in \bar{A}$ if and only if $\forall U \in \mathscr{T}$ such that $x \in U$, $U \cap A \neq \emptyset$.*
(2) *Similarly, $x \in \bar{A}$ if and only if $\forall B \in \mathscr{B}$ such that $x \in B$, $B \cap A \neq \emptyset$.*

Proof. (1) The statement of (1) is logically equivalent to $x \notin \bar{A}$ if and only if $\exists U \in \mathscr{T}$ such that $x \in U$ with $U \cap A = \emptyset$. Supposing $x \notin \bar{A}$, then $x \in U = X - \bar{A}$. As $\bar{A}$ is closed, we have found the desired open set U such that $A \cap U = \emptyset$. Conversely, supposing $U \subseteq \mathscr{T}$ is such that $x \in U$ and $A \cap U = \emptyset$, then $C = X - U$ is a closed set not containing x. As $\bar{A}$ is the intersection of all closed sets containing A, it follows that $x \notin \bar{A}$.

(2) The second statement follows from the first. If $x \in \bar{A}$, by (1) all open sets of X containing x have nonempty intersection with A. In particular, all $B \in \mathscr{B}$ are open. Conversely, if every basic open set B containing x is such that $B \cap A \neq \emptyset$ and

U is an open set containing x, then there exists $x \in B \subseteq U$ implying there is some $y \in B \cap A \subseteq U \cap A$. Hence, $U \cap A \neq \emptyset$ for all such U and by statement (1) $x \in \bar{A}$. □

Example 7.13. On the real line, the set $A = (0, 1]$ has closure $\bar{A} = [0, 1]$. By Theorem 7.12, $0 \in \bar{A}$ as every open interval (basic open set) containing 0, contains some $0 < x < 1$ hence $x \in A$. For $y < 0$ or $y > 1$ we find the open interval $(y-1, y/2)$ and $(\frac{y+1}{2}, y+1)$ respectively contain y but do not intersect A. Hence $\bar{A}$ is precisely the closed interval $[0, 1]$.

Example 7.14. For the closure of the rationals, $\bar{\mathbb{Q}} = \mathbb{R}$. Let x be any real number and let $a < x < b$. We can find some integer n, such that $nb - na > 1$. Then there is some integer $k \in (na, nb)$. Then $\frac{k}{n} \in (a, b)$. Thus $(a, b) \cap \mathbb{Q}$ is non-empty, implying $x \in \bar{\mathbb{Q}}$.

Example 7.15. In the set $A = (0, 1]$ from Example 7.13, the set $B = (0, \frac{1}{2})$ is a subset of A. Its closure in $\mathbb{R}$ is $\bar{B} = [0, \frac{1}{2}]$; by Theorem 7.11, its closure in A is $[0, \frac{1}{2}] \cap Y = (0, \frac{1}{2}]$.

Obviously $A \subseteq \bar{A}$, so the point of Theorem 7.12 is to give us a tool to find the remaining points of $\bar{A}$. What we are really after here are limit points of A.

7.4. **Limit Points.**

Definition 7.16. Let A be a subset of a topological space X. A point $x \in X$ is a *limit point* of A if every open set U containing x contains some point of A different from x. In symbols: $U \cap (A - \{x\}) \neq \emptyset$.

Example 7.17. Let $A = (0, 1]$ as in Example 7.13. The point 0 is a limit point of A as is every point of A, but no other points in $\mathbb{R}$.

One use for limit points is to help construct closures of sets.

Theorem 7.18. *Let A be a subset of a topological space X and denote the set of limit points of A by A'. Then $\bar{A} = A \cup A'$.*

Proof. The proof that $\bar{A} = A \cup A'$ follows almost immediately from Theorem 7.12.

($\bar{A} \subseteq A \cup A'$) Let $x \in \bar{A}$. If $x \in A$ we are done, so suppose $x \notin A$. From Theorem 7.12, $x \in \bar{A}$ if and only if for every open set U containing x, $U \cup A \neq \emptyset$. As $x \notin U \cup A$, there exists some other point $y \in U \cup A$. Hence $x \in A'$ and we conclude that $\bar{A} \subseteq A \cup A'$.

($A \cup A' \subseteq A'$) We know that $A \subseteq \bar{A}$, so it is left to show that $A' \subseteq \bar{A}$. Let $x \in A'$ be given. By the definition of limit points, for any open set U containing x the intersection $U \cap A$ contains a point other than x, in particular it is non-empty. Thus by Theorem 7.12, $x \in \bar{A}$. Thus $A' \subseteq \bar{A}$ and we conclude that $A \cup A' \subseteq \bar{A}$. □

The following corollary is a nice resulting characterization of closed sets that finds much practical use in space that are familiar to us (those where limits are relatively easy to compute).

Corollary 7.19. *A subset of a topological space is closed if and only if it contains its limit points.*

As in Theorem 7.12 we are able to replace the open sets in Definition 7.16 with basic open sets. We are just a careful reformulation away from redefining limits of sequences from calculus.

Example 7.20. Let $f : \mathbb{Z}_+ \to \mathbb{R}$ be some function, and denote $f(n)$ by x_n. Then $L \in \mathbb{R}$ is a limit point of the set $\text{Im}(f) = \{x_n\}_n$ if and only if $\forall \epsilon > 0 \exists x_n \in \text{Im}(f)$ such that $x_n \in (L-\epsilon, L+\epsilon)$. This particular collection of open intervals is sufficient as each open interval $(L-\epsilon, L+\epsilon)$ is a basis element of the usual basis for $\mathbb{R}$ and each open interval containing L contains $(L-\epsilon, L+\epsilon)$ as a subset for some ϵ.

For a given ϵ, if there is only a finite collection $A = \{x_n\}$ such that each $x_n \in (L-\epsilon, L+\epsilon)$ then A has a maximal element x_k implying for $\epsilon' = \frac{|L-x_k|}{2}$, $\text{Im}(f) \cap (L-\epsilon', L+\epsilon') = \emptyset$, contradicting that L is a limit point of $\text{Im}(f)$. Thus for each ϵ there exists infinitely many $n \in \mathbb{Z}_+$ such that $x_n \in (L-\epsilon, L+\epsilon)$.

Therefore the set $\{n \in \mathbb{Z}_+ : x_n \notin (L-\epsilon, L+\epsilon)\}$ is finite, implying it has a maximal element N. That is for all $n > N$, $x_n \in (L-\epsilon, L+\epsilon)$, and we have recovered the limit definition from calculus: L is a limit of the sequence $\{x_n\}$ if and only if $\forall \epsilon > 0 \, (\exists N \in \mathbb{Z}_+ (n > N \Rightarrow x_n \in (L-\epsilon, L+\epsilon)))$. The inclusion is typically written as $|L - x_n| < \epsilon$.

This notion of limits of sequences extends beyond calculus for us now, we replace $(L-\epsilon, L+\epsilon)$ with basic open sets.

Definition 7.21. Let $f : \mathbb{Z}_+ \to \mathbb{X}$ be a function defining the sequence $\{x_n = f(n)\}$. We say that the sequence $\{x_n\}$ *converges to the point* $x \in X$ if for all open sets U containing x there exists $N \in \mathbb{Z}_+$ such that for all $n > N$, $x_n \in U$.

Example 7.22. Let $X = \mathbb{R} \cup \{\mathcal{O}\}$ be given the topology generated by the basis

$$\mathscr{B} = \{(a,b) \subseteq \mathbb{R} : a < b\} \cup \{(a,0) \cup \{\mathcal{O}\} \cup (0,b) : a < 0 < b\}.$$

This space is called the line with two origins. Define the sequence $A = \{x_n \in X : x_n = \frac{1}{n}\}$. A converges to exactly two points in X: 0 and $\mathcal{O}$. This differs significantly from the real numbers, where if a sequence converges, it converges uniquely.

Example 7.23. Let $X = \{a, b, c\}$ and let $\mathscr{T}$ be the topology on X given by

$$\mathscr{T} = \{\emptyset, X, \{a,b\}, \{b\}, \{b,c\}\}.$$

Let $\{x_n\}$ be the constant sequence given by $x_n = b$ for all $n \in \mathbb{Z}_+$. Then $\{x_n\}$ converges to b as expect but also converges to a and c.

The unusual behavior of the limits in Examples 7.22 and 7.23 comes in two levels. In the line with two origins, every open set containing $\mathcal{O}$ and every open set containing 0 have non-empty intersection. There are still open sets containing one of $\mathcal{O}$ and 0 but not the other. The topological space in Example 7.23 does not have this property, rather, every open set containing c contains b.

The issue is best illustrated by the singletons. Most topologies that we have used throughout our careers have been such that singletons are closed. In Example 7.23, the singleton $\{b\}$ is not closed. What we have observed here are explained by the separation axioms.

7.5. **Separation Axioms.** The separation axioms are treated with more depth in Chapter 4 of Munkres but we will briefly take a look at two here.

Definition 7.24. Let $X, \mathscr{T}$ be a topological space.

(1) If for all distinct pairs $x, y \in X$ there exists $U, V \in \mathscr{T}$ such that $x \in U, y \in V$ and $x \notin V, y \notin U$, then X is said to be a T_1 *space*.
(2) If for all distinct pairs $x, y \in X$ there exists $U, V \in \mathscr{T}$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$, then X is said to be a T_2 *space*.

T_2 spaces are more frequently called *Hausdorff spaces* after Felix Hausdorff who first recognized their importance in imparting the useful properties in geometry and analysis of which we take advantage. They are also called *separated spaces*, the open sets U, V separating the pair x, y.

What baring do the separation axioms have on closed sets? Well:

Theorem 7.25. *Every finite point set in a T_1 space is closed.*

Proof. Let X be a T_1 space and let the point $x \in X$ be given. The T_1 axioms implies for every $y \in X$ there exists an open set V_y such that $y \in V_y$ and $x \notin V_y$. Then $X - \{x\} = \bigcup_{y \neq x} V_y$ is a union of open sets and hence open. Thus the singleton $\{x\}$ is closed. As the finite union of closed sets is closed, all finite point sets are closed in X. □

In fact Theorem 7.25 is an equivalence. We further get the following corollary for Hausdorff spaces:

Theorem 7.26. *Every finite point set in a Hausdorff space is closed.*

Proof. Every Hausdorff space is necessarily also a T_1 space. The conclusion follows from Theorem 7.25. □

Returning to our motivation for the separation axioms, lets look at how limits behave in T_1 and T_2 spaces.

Theorem 7.27. *Let X be a T_1 space and let A be a subset of X. Then the point x is a limit point of A if and only if every open set containing x contains infinitely many points of A distinct from x.*

Proof. If every open set containing x contains infinitely many points of A, then x is a limit point of A by definition.

We prove the converse by way of contradiction. Suppose x is a limit point of A and let U be an open set such that $x \in U$ and $U \cap A$ is a finite set. Notice $U \cap (A - \{x\}) = \{x_1, \ldots x_n\}$ is also finite. By Theorem 7.25, $\{x_1, \ldots x_n\}$ is closed and $X - \{x_1, \ldots x_n\}$ is open. Then $U \cap (X - \{x_1, \ldots x_n\})$ is an open set containing x with $[U \cap (X - \{x_1, \ldots x_n\})] \cap (A - \{x\}) = \emptyset$ contradicting that x is a limit point of A. Thus our supposition that there exists an open set U containing x meeting A in a finite number of points must be false, completing the proof by contradiction. □

This result seems strong, particularly where finite sets X are concerned. Does Theorem 7.27 say that finite sets cannot be T_1? No, but it does say that in a finite T_1 the set of limit points for a given set is always empty. For that matter:

Corollary 7.28. *Let $X, \mathscr{T}$ be a finite T_1 space. Then $\mathscr{T}$ is the discrete topology on X.*

Proof. Let A be a subset of X. By Theorem 7.27, the set of limit points of A is empty. By Theorem 7.18, $\bar{A} = A$. As the closure is a closed set, A must be closed. Thus all subsets of X are closed and hence all subsets are open. Thus the topology on X must be the discrete topology. □

We are one step away from fixing any strange limit behavior. Limit points in Hausdorff spaces are even more well behaved.

Theorem 7.29. *Let X be a Hausdorff space and let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to some point $x \in X$, then that point is unique. I.e. if $y \neq x$ then $\{x_n\}$ does not converge to y.*

Proof. Suppose $\{x_n\}$ converges to the point $x \in X$. Let $y \neq x$ be given. As X is assumed to be a Hausdorff space, there exists open sets U, V such that $x \in U$ and $y \in V$ with $U \cap V = \emptyset$. As $\{x_n\}$ converges to x, there exists $N \in \mathbb{Z}_+$ such that $x_n \in U$ for $n > N$. Then there is only a finite number, say k, of $\{x_n\}$ that are not in U. Thus $|V \cap \{x_n\}| \leq k$. Hence the sequence does not converge to y. □

As convergence is unique in Hausdorff spaces, it is justified to write $x_n \to x$ to mean x_n converges to x, or x is a limit of the sequence $\{x_n\}$.

Finally, to see how well behaved the Hausdorff axiom is when interacting with our earlier studied properties, we have the following theorem:

Theorem 7.30. *Every set with a strict total order is a Hausdorff space in the order topology. The product space of two Hausdorff spaces satisfies the Hausdorff axiom. A subspace of a Hausdorff space satisfies the Hausdorff axiom.*

8. Continuous Functions

8.1. **Introduction.** We previously discussed how functions are the objects of mathematics used to name elements, distinguish objects, measure properties, determine sameness. We will now build this into our study of topological spaces. The ideas of this section have their foundation in the concepts of continuity found in calculus. Of course, we want to generalize these notions so that they depend only the given function and the concept of open sets.

8.2. **Continuity.**

Definition 8.1. Let X and Y be topological spaces. A function $f : X \to Y$ is *continuous* if for each open subset V of Y, the preimage $f^{-1}(V)$ is open in X.

We remember this definition as "the preimage of open sets are always open." As we would expect (or at least hope), in order to prove a function is continuous we need only show the preimage of basic open sets are open.

Lemma 8.2. *Let X and Y be topological spaces and let $\mathscr{B}_Y$ be a basis for the topology on Y. The a function $f : X \to Y$ is continuous if and only if for all $B \in \mathscr{B}_Y$ the preimage $f^{-1}(B)$ is open in X.*

Proof. Let X, Y be given. Suppose $f : X \to Y$ is continuous. If B is a basis element of $\mathscr{B}_Y$ a basis for the topology on Y, then B is open in Y. By continuity, $f^{-1}(B)$ is open in X.

Conversely, suppose that for every basic open set $B \in \mathscr{B}_Y$, the preimage $f^{-1}(B)$ is open in X and let V be an arbitrary open set in Y. As $\mathscr{B}_Y$ is a basis, by Lemma 5.6 $V = \bigcup B_\alpha$ for some collection of basic open sets $\{B_\alpha\}$. As preimages of unions are the unions of preimages, $f^{-1}(V) = f^{-1}(\bigcup B_\alpha) = \bigcup f^{-1}(B_\alpha)$. Hence $f^{-1}(V)$ is the union of open sets which is open in X. □

This result can be broken down even further: a function is continuous if and only if the preimage of subbasis elements are open.

Example 8.3. Lemma 8.2 implies that the ε-δ definition of continuity from calculus is equivalent to our own when $\mathbb{R}$ is considered with the usual topology.

Example 8.4. When we go beyond the topology on $\mathbb{R}$ that we are accustomed to, continuous functions immediately behave differently. The function $f : \mathbb{R} \to \mathbb{R}_l$ given by the rule $f(x) = x$ is not continuous. The preimage of $[0, 1)$ from $\mathbb{R}_l$ is the set $f^{-1}([0, 1)) = [0, 1)$ which is not open in the usual topology.

On the other hand the function $g : \mathbb{R}_l \to \mathbb{R}$ given by $g(x) = x$ is continuous as the preimage of (a, b) is the set $f^{-1}((a, b)) = (a, b)$ which is open in $\mathbb{R}_l$. Notice we immediately make use of Lemma 8.2: we only consider basic open sets to determine continuity.

We can determine continuity a number of other ways as well.

Definition 8.5. Let X and Y be topological spaces and let $f : X \to Y$ be given. We say f is *continuous at the point $x \in X$* if for each open set V containing $f(x)$, there is an open set U containing x such that $f(U) \subseteq V$.

Theorem 8.6. *Let X and Y be topological spaces and let $f : X \to Y$ be given. The following are equivalent:*

(1) *f is continuous.*
(2) *For every subset A of X, $f(\bar{A}) \subseteq \overline{f(A)}$.*
(3) *For every closed set B of Y, the preimage $f^{-1}(B)$ is closed in X.*
(4) *For all $x \in X$, f is continuous at the point x.*

Proof. We first show that (1) and (4) are equivalent. Suppose f is continuous and let $x \in X$. Let V be an open set in Y containing $f(x)$. As f is continuous $U = f^{-1}(V)$ is open, and by an exercise from Section 2 $f(U) \subseteq V$. Thus f is continuous at x for all $x \in X$. Conversely, suppose f is continuous at each point $x \in X$, and let V be an open set in Y. If $V \cap f(X) = \emptyset$ then the preimage of V is empty and hence open in X. Then let $V \cap f(X)$ be nonempty and let $x \in f^{-1}(V)$. Then $f(x) \in V$ and by supposition there exists some U_x open in X containing x such that $f(U_x) \subseteq V$. Then $U_x \subseteq f^{-1}(V)$ and $f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$ is open. Thus (1) $\iff$ (4).

(1)$\Rightarrow$(2). Assume f is continuous and let A be a subset of X. Let $x \in \bar{A}$. Then $f(x) \in f(\bar{A})$. Let V be an open set containing $f(x)$, by (4) there exists an open set U containing x such that $f(U) \subseteq V$. As U contains x and $x \in \bar{A}$, $U \cap A \neq \emptyset$. In particular, there exists $y \in U \cap A$ and thus

$$f(y) \in f(U \cap A) \subseteq f(U) \cap f(A) \subseteq V \cap f(A).$$

Thus every open set V containing $f(x)$ has non-empty intersection with $f(A)$ and by definition $f(x) \in \overline{f(A)}$. As every point of $f(\bar{A})$ can be written as $f(x)$ for some $x \in \bar{A}$, we have shown $f(\bar{A}) \subseteq \overline{f(A)}$.

(2)$\Rightarrow$(3). Suppose for every subset A of X, $f(\bar{A}) \subseteq \overline{f(A)}$ and let B be a closed subset of Y. Fix $A = f^{-1}(B)$, we aim to show $\bar{A} = A$ implying A is closed. As $A \subseteq \bar{A}$ be default, we need only show $\bar{A} \subseteq A$. We know that $f(A) = f(f^{-1}(B)) \subseteq B$. Thus $\overline{f(A)} \subseteq \bar{B} = B$. We now have

$$f(x) \in f(\bar{A}) \subseteq \overline{f(\bar{A})} \subseteq B,$$

hence $x \in f^{-1}(B) = A$. Thus $\bar{A} \subseteq A$ implying $A = \bar{A}$ as desired.

(3)$\Rightarrow$(1). Suppose the preimage of every closed set in Y is closed in X and let V be an open set in Y. Then $Y - V$ is closed and by supposition $f^{-1}(Y - V)$ is closed in X. Then $f^{-1}(Y - V) = f^{-1}(Y) - f^{-1}(V) = X - f^{-1}(V)$ is closed implying $f^{-1}(V)$ is open in X. □

8.3. **Homeomorphisms.** We now use the notion of continuity to develop the notion of sameness in the study of topology.

Definition 8.7. Let X and Y be topological spaces and let $f : X \to Y$ be a bijection. If both f and $f^{-1} : Y \to X$ are continuous, then f is called a *homeomorphism*. The spaces X and Y are said to be *homeomorphic*.

By exchanging f and f^{-1} we see that f^{-1} also satisfies the definition of being a homeomorphism. As f^{-1} is continuous, the preimage of an open set U in X is open in Y. But the preimage of under f^{-1} is the image under f. Thus a set U is open in X if and only if $f(U)$ is open in Y; similarly, V is open in Y if and only if $f^{-1}(Y)$ is

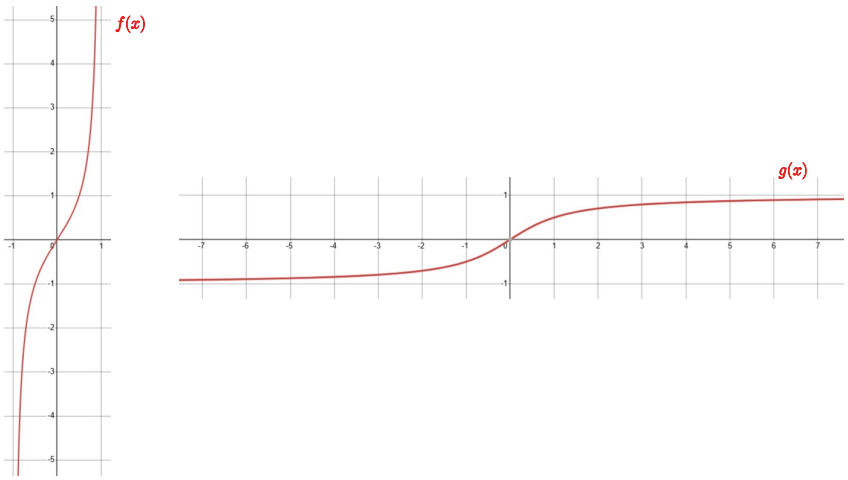


FIGURE 2. The function $f(x) = \tan(\frac{\pi x}{2})$ and its inverse $g(x) = \frac{2}{\pi}\arctan(x)$

open in X. Thus a homeomorphism is not only a bijection between X and Y, but it also gives a bijection between the collection of open sets of X and the collection of open sets of Y. In this way, we think of the topological spaces X and Y as being the same space and the bijection $f : X \to Y$ can be thought of as copying the topology of X onto Y.

As a result, any property of X or of a subset $A \subseteq X$ corresponds through f to a property of Y or of $f(A) \subseteq Y$. Such properties are called *topological properties*. One of the primary tasks in topology is to decide if two topological spaces X, Y are the same, i.e. homeomorphic. If X has some topological property but Y does not... we have determined that X and Y are not homeomorphic.

Definition 8.8. Let $f : X \to Y$ be an injective, continuous function and let $Z = f(X)$. The $f' : X \to Z$ given by $f'(x) = f(x)$ is a continuous bijection. If f' happens to be a homeomorphism, we say that f is an *embedding* of X into Y.

Example 8.9. The function $f : (-1, 1) \to \mathbb{R}$ given by $f(x) = \tan(\frac{\pi x}{2})$ is a homeomorphism. It is clearly a bijection, and by Exercise 1 of the Section 18 Homework is continuous, being an elementary function. Likewise, it's inverse $g : \mathbb{R} \to (-1, 1)$ given by $g(x) = \frac{2}{\pi}\arctan(x)$ is continuous. The graphs of f and g are shown in Figure 2.

Example 8.10. The unit circle in $\mathbb{R}^2$ can be represented in polar coordinates as

$$S^1 = \{(r, \theta) : r = 1 \wedge \theta \in [0, 2\pi)\}.$$

We give S^1 the subspace topology in $\mathbb{R}^2$. The function $f : S^1 \to \mathbb{R}^3$ given by $f(r, \theta) = (\sin(\theta) + 2\sin(2\theta), \cos(\theta) - 2\cos(2\theta), -\sin(3\theta))$ is an embedding of S^1 into $\mathbb{R}^3$. The image of f is shown in Figure 3.

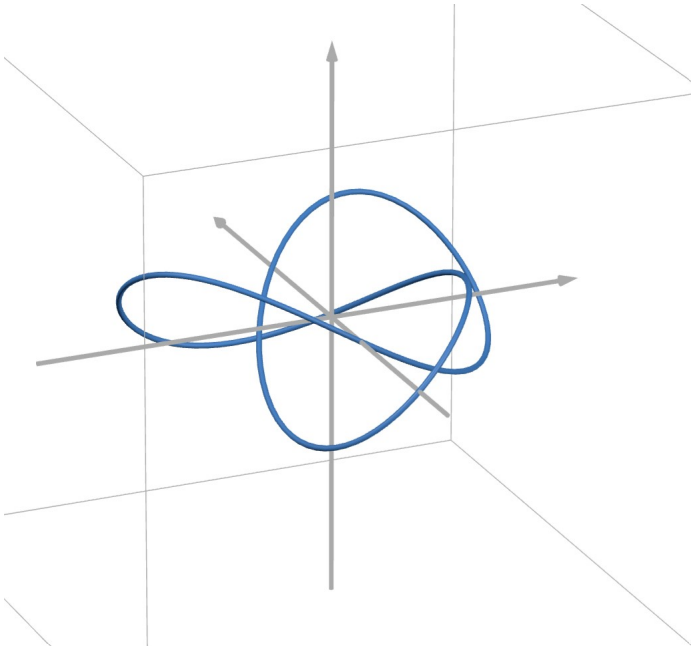


FIGURE 3. The image of f in $\mathbb{R}^3$. This subset of $\mathbb{R}^3$ is commonly called the trefoil knot.

8.4. **Constructing Continuous Functions.** In analog with the techniques of constructing new functions found in Section 2, we look at how these constructions interact with the notion of continuity.

Theorem 8.11. *Let X, Y, Z be topological spaces.*

(1) *If $f : X \to Y$ is given by the rule $f(x) = y$ for some fixed $y \in Y$, then f is continuous.*
(2) *If A is a subspace of X, the inclusion map $\iota : A \to X$ is continuous. In fact, it must be an embedding.*
(3) *If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous.*
(4) *If $f : X \to Y$ is continuous and A a subspace of X, then the restriction $f|_A : A \to Y$ is continuous.*
(5) *Let $f : X \to Y$ be continuous. For Z a subspace of Y containing the image $f(X)$, then the function $g : X \to Z$ given by $g(x) = f(x) \in Z \subseteq Y$ is continuous. Similarly, if Y is a subspace of Z, then $h : X \to Z$ given by $h(x) = f(x) \in Y \subseteq Z$ is continuous.*
(6) *The function $f : X \to Y$ is continuous if X can be written as the union of open sets U_α such that for each α, $f|_{U_\alpha} : U_\alpha \to Y$ is continuous.*

Theorem 8.12 (The pasting lemma)**.** *Let $X = A \cup B$, where A, B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous, where A, B are given the subspace topologies. If $f(x) = g(x)$ for all $x \in A \cap B$, then the function $h : X \to Y$ given by*

$$h(x) = \begin{cases} f(x) & , x \in A \\ g(x) & , x \in B \end{cases}$$

is continuous.

Proof. We use Theorem 8.6 (3), the equivalent condition for continuity. Let C be closed in Y. Then the preimage $h^{-1}(C)$ is found to be

$$\begin{aligned} h^{-1}(C) &= h^{-1}(C \cap X) \\ &= h^{-1}(C \cap (A \cup B)) \\ &= h^{-1}((C \cap A) \cup (C \cap B)) \\ &= h^{-1}(C \cap A) \cup h^{-1}(C \cap B) \\ &= f^{-1}(C) \cup g^{-1}(C) \end{aligned}$$

As f, g are continuous, the preimages are closed. Since the finite union of closed sets are closed, $h^{-1}(C)$ is closed. Hence h is continuous. □

The pasting lemma is not so unfamiliar, it is the reason for piecewise defined function in calculus we check continuity of each piece and ensure the pieces are equal on the overlap (we usually add the overlap ourselves).

Example 8.13. The piecewise defined function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} e^x & , x \leq 0 \\ x + 1 & , x > 0 \end{cases}.$$

To apply the pasting lemma directly we check that $e^0 = 0 + 1$ and note that

$$g(x) = \begin{cases} e^x & , x \leq 0 \\ x + 1 & , x \geq 0 \end{cases}$$

defines the same function as f. By the pasting lemma, g is continuous hence f is continuous. Of course, this is over doing it, the simple check of equality where the closure of the domains intersect suffices to show f is continuous.

Another check for continuity that we have seen in calculus is given for functions whose codomains are Cartesian products.

Theorem 8.14 (Functions into products)**.** *Let $f : A \to X \times Y$ be given by the rule $f(a) = (f_1(a), f_2(a))$. Then f is continuous if and only if the functions $f_1 : A \to X$ and $f_2 : A \to Y$ are continuous.*

Proof. First note that the function $f : A \to X \times Y$ can always be written with a rule of this form: Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be the projection maps. These maps are continuous and

$$\pi_1(f(a)) = f_1(a) \quad \text{and} \quad \pi_2(f(a)) = f_2(a)$$

.

If f is continuous, then f_1 and f_2 are composites of continuous functions are themselves continuous. Suppose that f_1 and f_2 are continuous; we will conclude that f is continuous. Let $U \times V$ be a basic open set in $X \times Y$. A given point a is in $f^{-1}(U \times V)$ if and only if $f(a) \in U \times V$ if and only if $f_1(a) \in U \wedge f_2(a) \in V$. Thus $f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V)$ is the intersection of open sets and thus open. Therefore we conclude that f is open. $\square$

9. The Metric Topology

9.1. **Introduction.** In this section we look at another way to generate the standard topology on $\mathbb{R}^n$ using a metric. As we did with the order topology, we will consider how this topology generalizes to arbitrary spaces for which we can define metrics. For $\mathbb{R}^n$, many typical metrics generate the standard topology and the notions of continuity, open/closed sets, and limits are exactly as we find them in calculus, not mere consequences of our, perhaps seemingly arbitrary, definitions seen thus far.

One take away from this section is how truly special Euclidean space is. The way in which continuous functions behave, for example, is particularly nice. This nicety while perhaps not immediately apparent is a consequence of some of the ideas we will establish in this section. This mirrors what we saw in Example 7.22, where limits in arbitrary spaces may behave in ways one would not consider nice. The issue of non-unique convergence of sequences was resolved by the Hausdorff property. We will now get a more complete picture (pun intended).

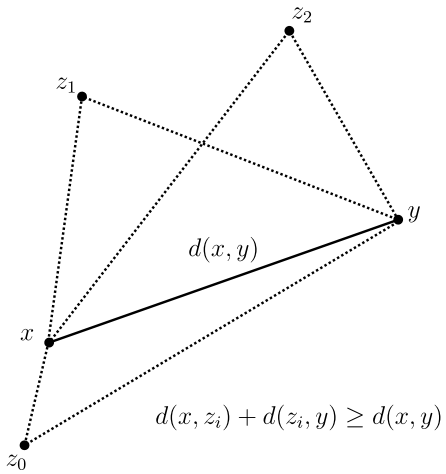
9.2. **Metrics.** The reader has likely encountered metrics at several points in their studies. A metric is a formalization of the notion of distance: the distance between points in Euclidean space is the obvious example, but normed vector spaces also have this notion of distance. Little did we know when learning linear algebra, but we were actually studying topological spaces!

Definition 9.1. Let X be a set. The function $d : X \times X \to \mathbb{R}$ is called a *metric* if the following properties hold:

(1) $d(x, y) \geq 0$ for all $x, y \in X$; with equality if and only if $x = y$.
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$.
(3) (**The Triangle Inequality**) $d(x, z) + d(z, y) \geq d(x, y)$ for all $x, y, z \in X$.

For points $x, y \in X$ we call $d(x, y)$ the *distance* between x and y in the metric d. In more general terms d is a "real" metric as it takes on values from $\mathbb{R}$ as opposed to $\mathbb{C}$ or some other field of scalars.

The Triangle Inequality gets its name from the Euclidean plane where the shortest distance between two points is a straight line (a geodesic) illustrated in the figure below. The reader is of coursed cautioned not to think that every metric space has geodesics.



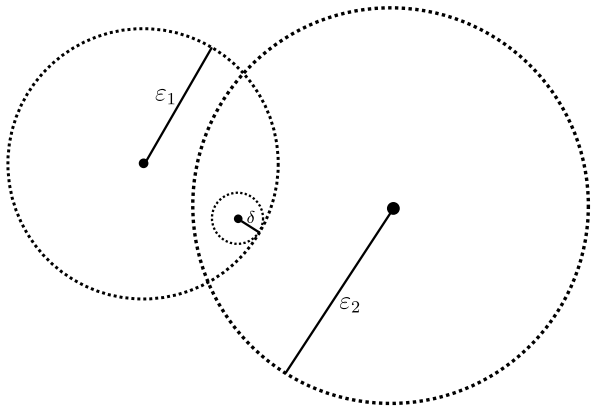
Definition 9.2. Given some $\varepsilon > 0$, the set

$$B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$$

is called the *open ball* of radius ε centered at x. It is also called the *ε-ball*.

Definition 9.3. If d is a metric on X, then the collection of all open balls for a basis for a topology on X called the *metric topology* induced by d.

Checking that this collection is a basis is straightforward. The first basis criterion is satisfied by property (1) of the definition of a metric: $d(x, x) = 0 < \varepsilon$ thus $x \in B_d(x, \varepsilon)$ for any ε. For the second criterion, for $y \in B_d(x_1, \varepsilon_1) \cap B_d(x_2, \varepsilon_2)$ setting $\delta = \min\{\varepsilon_i - d(x_i, y)\}$ we have $y \in B_d(y, \delta) \subseteq B_d(x_1, \varepsilon_1) \cap B_d(x_2, \varepsilon_2)$.



By Lemma 5.3 we can then describe a set U in a metric topology as open if and only if for all $x \in U$ there exists some δ such that $B_d(x, \delta) \subseteq U$.

Example 9.4. The standard metric on $\mathbb{R}$ is defined by $d(x, y) = |x - y|$. The open balls are then $B_d(x, \varepsilon) = (x - \varepsilon, x + \varepsilon)$. It was already shown in the Section 18 Homework that the collection of all such open balls generates the standard topology on $\mathbb{R}$.

Definition 9.5. A topological space X is said to be *metrizable* if there exists a metric $d : X \times X \to \mathbb{R}$ that induces the same topology on X. The space X together with the metric d are called a *metric space*.

Definition 9.6. Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ be given. We define the *norm* of x by

$$||x|| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2};$$

we define the *Euclidean metric* on $\mathbb{R}^n$ by

$$d(x, y) = ||x - y||.$$

We define the *square metric* ρ by

$$\rho(x, y) = \max_{1 \le i \le n} \{|x_i - y_i|\}.$$

In $\mathbb{R}$, the two metrics coincide as

$$d(x, y) = \sqrt{(x - y)^2} = |x - y| = \max_{1 \le i \le 1} \{|x_i - y_i|\} = \rho(x, y).$$

In $\mathbb{R}^2$, the open balls $B_d(x,\varepsilon)$ in the metric topology look like circular regions centered at x with radius ε, the inspiration for these definition. The open balls $B_\rho(x,\varepsilon)$ are square regions with side length 2ε and x at the intersection of the two diagonals. The visualization extends to $\mathbb{R}^n$ into balls and cubes of dimension n.

Despite our familiarity and intuition surrounding the Euclidean metric, it will be more convenient for us to use the square metric. After all, basic open sets in the standard topology on $\mathbb{R}^n$ are already "boxes," being of the form $B = (a_1,b_1) \times (a_2,b_2) \times \cdots \times (a_n,b_n)$ where (x_i,y_i) denotes an open interval in $\mathbb{R}$. Nevertheless, both d and ρ generate the same topology on $\mathbb{R}^n$ and it is the standard topology as we have already defined.

Lemma 9.7. *Let d and d' be two metrics on the set X; denote the topologies generated by d, d' by $\mathscr{T}, \mathscr{T}'$ respectively. Then $\mathscr{T}'$ is finer than $\mathscr{T}$ if and only if for each $x \in X$ and each $\varepsilon >$, there exists a $\delta > 0$ such that*

$$B_{d'}(x,\delta) \subseteq B_d(x,\varepsilon).$$

Proof. The proof follows from Lemma 5.8 and the fact that the collection of open balls under d and d' form bases for $\mathscr{T}$ and $\mathscr{T}'$. □

Theorem 9.8. *The topologies on $\mathbb{R}^n$ induced by the Euclidean metric and the square metric are both the same as the product topology on $\mathbb{R}^n$.*

Proof. Let $x \in \mathbb{R}^n$ and let $\varepsilon > 0$ be given. Denote by d and ρ the Euclidean and square metrics respectively. Consider the inclusions $B_\rho(x, \frac{\varepsilon}{\sqrt{n}}) \subset B_d(x,\varepsilon) \subset B_\rho(x,\varepsilon)$. As $B_\rho(x, \frac{\varepsilon}{\sqrt{n}}) = (x_1-\varepsilon, x_1+\varepsilon) \times \cdots \times (x_n-\varepsilon, x_n+\varepsilon)$ is a basic open set in the standard topology on $\mathbb{R}^n$, by Lemmas 5.8 and 9.7 the standard topology is finer than the topology generated by both the Euclidean and square metrics.

Let $B = (a_1,b_1) \times (a_2,b_2) \times \cdots \times (a_n,b_n)$ be a basic open set in the standard topology on $\mathbb{R}^n$. For $x \in B$, set

$$\varepsilon = \min_{1\le i\le n} \min\{(b_i - x_i), (a_i - x_i)\}.$$

Then $B_d(x,\varepsilon) \subset B_\rho(x,\varepsilon) \subseteq B$. Again, by Lemma 5.8 and 9.7, the topologies generated by the Euclidean and square metrics are finer than the standard topology on $\mathbb{R}^n$.

We have then concluded that the three topologies on $\mathbb{R}^n$ are the same. □

We included Lemma 9.7 in the above proof only for instructive purposes when comparing the topologies generated by the two metrics. This lemma sees better applications when comparing two topologies for which the only basis available are those generated by the open balls.

9.3. **Properties of Metric Spaces.** We now consider how metric spaces behave in relation to our previously defined topological properties.

Proposition 9.9. *Let X be a metric space with metric d and let A be a subspace of X. Then function $d|_{A\times A} : A \times A \to \mathbb{R}$ is a metric on A such that the induced topology is the same as the subspace topology.*

Proof. Verifying that $d|_{A\times A}$ is a metric is left to the exercises. Assume $d' = d|_{A\times A}$ is a metric. The collection $\mathscr{C}$ of open balls $B_{d'}(a,\varepsilon)$ is a collection of open sets in the

subspace topology as $B_{d'}(a,\varepsilon) = B_d(a,\varepsilon) \cap A$ is the intersection of an open set in X with A. Let U be open in A and let $a \in U$ be given. By the definition of the subspace topology there exists a set V open in X such that $U = V \cap A$. Since the topology on X is generated by the open balls B_d, there exists some ε such that $B_d(a,\varepsilon) \subseteq V$, then $B_{d'}(a,\varepsilon) = B_d(a,\varepsilon) \cap A \subseteq V \cap A = U$.

Thus for each $a \in U$ there exists an element C of $\mathscr{C}$ such that $a \in C \subseteq U$. As $\mathscr{C}$ is a collection of open sets in A, $\mathscr{C}$ is thus a basis for the topology on A by Lemma 5.7. □

Proposition 9.10. *Let X and Y be metric spaces with metrics d_X and d_Y respectively. Then the product space $X \times Y$ is metrizable.*

Proof. This is analogous to the square matrix in $\mathbb{R}^n$. Let $\rho : (X \times Y) \times (X \times Y) \to \mathbb{R}$ be given by

$$\rho(x_1 \times y_1, x_2 \times y_2) = \max\{d_X(x_1,x_2), d_Y(y_1,y_2)\}.$$

Let us check that ρ is a metric on $X \times Y$:

(1) For all pairs $x_1 \times y_1$ and $x_2 \times y_2 \in X \times Y$, $\rho(x_1 \times y_1, x_2 \times y_2) \geq d_X(x_1,x_2) \geq 0$ and if $\rho(x_1 \times y_1, x_2 \times y_2) = 0$ then the maximum of $d_X(x_1,x_2)$ and $d_Y(y_1,y_2)$ is zero. As both d_X and d_Y assume only non-negative values. This implies $d_X(x_1,x_2) = 0$ and $d_Y(y_1,y_2) = 0$. Thus $x_1 = x_2$ and $y_1 = y_2$ as d_X and d_Y are metrics.

(2) Again for points $x_1 \times y_1$ and $x_2 \times y_2 \in X \times Y$

$$\begin{aligned}\rho(x_1 \times y_1, x_2 \times y_2) &= \max\{d_X(x_1,x_2), d_Y(y_1,y_2)\} \\ &= \max\{d_X(x_2,x_1), d_Y(y_2,y_1)\} \\ &= \rho(x_2 \times y_2, x_1 \times y_1).\end{aligned}$$

(3) For a third point $x_3 \times y_3$ we have

$$\begin{aligned}\rho(x_1 \times y_1, x_2 \times y_2) &= \max\{d_X(x_1,x_2), d_Y(y_1,y_2)\} \\ &\leq \max\{d_X(x_1,x_3) + d_X(x_3,x_2), d_Y(y_1,y_3) + d_Y(y_3,y_2)\} \\ &\leq \max\{d_X(x_1,x_3), d_Y(y_1,y_3)\} + \max\{d_X(x_3,x_2), d_Y(y_3,y_2)\} \\ &= \rho(x_1 \times y_1, x_3 \times y_3) + \rho(x_3 \times y_3, x_2 \times y_2).\end{aligned}$$

The proof that ρ induces the product topology on $X \times Y$ follows the proof of Theorem 9.8. □

A particularly nice consequence of metrizability is the Hausdorff property.

Proposition 9.11. *Every metric space is Hausdorff.*

Proof. Let X, d be a metric space and let x, y be distinct points in X. We will construct open sets U, V such that $x \in U, y \in V$, and $U \cap V = \emptyset$. Let $\delta = d(x,y)$. Since $x \neq y$, $\delta > 0$. We claim $U = B_d(x, \frac{\delta}{2}), V = B_d(y, \frac{\delta}{2})$ are the desired open sets. We have $x \in B_d(x, \frac{\delta}{2})$ be definition and likewise for y. Let $z \in U \cap V$. Then $d(x,z) < \frac{\delta}{2}$ and $d(z,y) < \frac{\delta}{2}$. But this implies $d(x,z) + d(z,y) < \delta = d(x,y)$ contradicting the Triangle Inequality. We conclude that there are no elements in the intersection $U \cap V$. As x, y were arbitrary, we have deduced that X must be Hausdorff. □

Turning our attention to order topologies, there is no direct relation to metric spaces. That is to say, not all spaces with the order topology are metrizable. Unfortunately an example of such behavior is hard to build here. The usual trick to showing a space is not metrizable is to show it is not Hausdorff, but as we have seen the order topology on any set with a strict total order is Hausdorff.

As for continuity of functions between metric spaces, we already have some intuition and it will turn out to be correct. It was shown in the homework that the ε-δ definition of continuity from calculus implied our new definition: preimages of open sets are open. While this was for functions $f : \mathbb{R} \to \mathbb{R}$, a similar results for any function between metric spaces.

Theorem 9.12. *Let X, d_X and Y, d_Y be metric spaces. The function $f : X \to Y$ is continuous if and only if for all $x \in X$ and all $\varepsilon > 0$ there exists $\delta > 0$ such that $d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$.*

Proof. We prove the biconditional in two parts:
($\Rightarrow$) Suppose f is continuous. As $B_{d_Y}(f(x), \varepsilon)$ is open, the preimage $f^{-1}(B_{d_Y}(f(x), \varepsilon))$ is open in X. As X is a metric space, openness implies for $x \in f^{-1}(B_{d_Y}(f(x), \varepsilon))$ there exists some open ball $B_{d_X}(x, \delta)$ such that $x \in B_{d_X}(x, \delta) \subseteq f^{-1}(B_{d_Y}(f(x), \varepsilon))$. If $d_X(x, y) < \delta$ then $y \in B_{d_X}(x, \delta)$, hence $f(y) \in B_{d_Y}(f(x)) < \varepsilon$. Thus implying $d_Y(f(x), f(y)) < \varepsilon$ as desired.

($\Leftarrow$) Suppose that for all $x \in X$ and all $\varepsilon > 0$ there exists $\delta > 0$ such that $d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$. We will show that f must be continuous. To this end, let V be open in Y and fix $x \in f^{-1}(V)$. As $f(x) \in V$ there exists some ε such that $B_{d_Y}(f(x), \varepsilon) \subseteq V$. By our hypothesis, there exists some $\delta > 0$ such that $d_X(x, y) < \delta$ implies $d_Y(f(x), f(y)) < \varepsilon$. That is $f(B_{d_X}(x, \delta)) \subseteq B_{d_Y}(f(x), \varepsilon)$ implying for x we have found an open set $B_{d_x}(x, \delta)$ such that $x \in B_{d_X}(x, \delta) \subseteq f^{-1}(V)$. We conclude that $f^{-1}(V)$ is open by definition in the metric space X. Thus f is continuous as the preimage of every open set is open. □

One property of $\mathbb{R}$ that sees great use in analysis is the equivalence of continuity to sequential convergence under a function's image. This is in fact a property shared by all metric spaces. To get there we need an intermediate result and a definition.

Definition 9.13. Let X and Y be topological spaces. A function $f : X \to Y$ is said to be *sequentially continuous* if for all convergent sequences $x_n \to x$ the sequence $\{f(x_n)\}$ converges to $f(x)$.

Lemma 9.14. *Let X be a topological space and let $A \subseteq X$. If if there is a sequence $\{a_n\} \subset A$ converging to a then $a \in \bar{A}$. The converse holds if X is metrizable.*

Proof. The main statement of the lemma is clear from the definition of convergence. As $a_n \to a$ then every neighborhood of a contains infinitely many a_n each of which is in A. Thus $a \in \bar{A}$.

As for the converse, suppose X is metrizable with metric d and let $a \in \bar{A}$ be some point. Consider for $n \in \mathbb{Z}_+$ the open ball $B_d(a, \frac{1}{n})$. In particular, as the open ball is open, there exists for each n some element of A distinct from a. Call this element a_n. We claim that the sequence $\{a_n\}$ so constructed converges to $a \in \bar{A}$. Let U be an open set in X containing a. As U is open there is some $\varepsilon > 0$ such that $B_d(a, \varepsilon) \subseteq U$.

Choose $N > \frac{1}{\varepsilon}$. Then for all $n > N$, $a_n \in B_d(a, \frac{1}{n}) \subseteq B_d(a, \varepsilon) \subseteq U$. We conclude that $a_n \to a$. □

Theorem 9.15. *Let X and Y be topological spaces. If the function $f : X \to Y$ is continuous, then f is sequentially continuous. The converse holds if X is metrizable.*

Proof. Again, the main statement is simple enough. If f is continuous and $x_n \to x$, the the preimage of an open set V containing $f(x)$ is open and by convergence contains all x_n for n greater than some determined $N \in \mathbb{Z}_+$. Thus $f(x_n) \in V$ for all $n > N$ implying convergence to $f(x)$.

For the converse, suppose X is metrizable and assume f is sequentially continuous. Let A be a subset of X. If $x \in \bar{A}$, then by Lemma 9.14 there is a sequence of $x_n \in A$ converging to x. As f is sequentially continuous $f(x_n) \to f(x)$. Since each $f(x) \in f(A)$ by Lemma 9.14, $f(x) \in \overline{f(A)}$. By definition of the image $f(x) \in f(\bar{A})$. Hence we have concluded that for all $A \subseteq X$, $f(\bar{A}) \subseteq \overline{f(A)}$. This is the second equivalent formulation of continuity in Theorem 8.6. □

There are two things to note about Theorem 9.15. First, we made no comment on the properties of the space Y, only X was required to be metrizable in the converse. Second, we did not actually use the assumption that X was metrizable, nor did we actually use it in the converse to Lemma 9.14. Instead, we used the countable collection $\{B_d(a, \frac{1}{n})\}$ to define our sequence a_n. The existence of such a collection is weaker than being metrizable (as we used in the proof, every metrizable space has such a collection).

Definition 9.16. A topological space X is said to have a *countable basis at the point* x if there is a countable collection $\{U_n\}$ of open sets each containing x such that for any open set V containing x there is at least one U_n such that $x \in U_n \subseteq V$. If X has a countable basis at each point, then X is said to satisfy the *first countability axiom* and is said to be *first countable*.

All metric spaces are first countable, and by the proofs of Lemma 9.14 and Theorem 9.15 we can replace the metrizable criterion for first countable and the result still holds. The first countability axiom along with the other notions of countability are covered in depth in Chapter 4 of Munkres.

Example 9.17. To see that the converse does not hold in general. We will find a space in which sequential continuity does not imply continuity. Let $\mathbb{R}_{\mathcal{C}}$ denote the reals with the cocountable topology $\mathscr{T}_c$. Convergent sequences in $\mathbb{R}_{\mathcal{C}}$ are certainly odd; let $x_n \to x$. Define

$$A = \{x_n : x_n \neq x\}.$$

The set A is of course countable, hence closed in $\mathbb{R}_{\mathcal{C}}$. Suppose for all $N \in \mathbb{Z}_+$ that there is some $n > N$ such that $x_n \in A$. Then $U = \mathbb{R}_{\mathcal{C}} - A$ is an open set containing x such that $x_n \notin U$ for arbitrarily large n. Hence the sequence $\{x_n\}$ does not converge to x, a contradiction.

We can conclude that there must exist some N such that for all $n > N$, $x_n \notin A$. In other words $x_n = x$. We call such sequences eventually constant. As all eventually constant sequences converge (to said constant) we have shown that, in fact, the only convergent sequences in $\mathbb{R}_{\mathcal{C}}$ are eventually constant.

The implication is clear: $f : \mathbb{R}_{\mathcal{C}} \to \mathbb{R}$ given by $f(x) = x$ is sequentially continuous but is not continuous (the preimage of $(0, 1)$ is $(0, 1)$ which is not open in $\mathbb{R}_{\mathcal{C}}$).

Now by Theorems 9.12 and 9.15, all of what we know about functions from calculus and analysis hold here for metric spaces.

Theorem 9.18. *The addition, subtraction, and multiplication operations are continuous functions from $\mathbb{R}^2$ to $\mathbb{R}$. The division operation is a continuous function form $\mathbb{R} \times (\mathbb{R} - \{0\})$ to $\mathbb{R}$.*

The sum and difference of two real valued functions depends only on the real outputs. Thus we have for an arbitrary topological space:

Theorem 9.19. *Let X be a topological space. If $f, g : X \to \mathbb{R}$ are continuous functions, then $f + g$, $f - g$, and $f \cdot g$ are continuous. If $g(x) \neq 0$ for all x, then f/g is continuous.*

Proof. The function $h : X \to \mathbb{R} \times \mathbb{R}$ defined by $h(x) = (f(x), g(x))$ is continuous by Theorem 8.11. The functions $f+g$, $f-g$, and $f \cdot g$ are compositions of the arithmetic operations and h and as the composition of continuous functions are continuous. The quotient f/g for $g(x) \neq 0$ for all x is the composition of the division operation (continuous on $\mathbb{R} \times (\mathbb{R} - \{0\})$) with the restriction of the codomain of $h : X \to \mathbb{R} \times (\mathbb{R} - \{0\})$ which is here well defined by assumption on the image of g. By Theorem 8.11 f/g is continuous. □

Definition 9.20. Let X be a topological space and Y, d a metric space. Let $f_n : X \to Y$ be a sequence of functions. We say $\{f_n\}$ *converges uniformly* to the function $f : X \to Y$ if given $\varepsilon > 0$ there exists $N \in \mathbb{Z}_+$ such that $d(f_n(x), f(x)) < \varepsilon$ for all $n > N$ and all $x \in X$.

Theorem 9.21 (The Uniform Limit Theorem)**.** *Let $f_n : X \to Y$ be a sequence of continuous functions where Y, d is a metric space. If $\{f_n\}$ converges uniformly to f, then f is continuous.*

Proof. We will show that f is continuous at $x \in X$ for all $x \in X$. Fix $x_0 \in X$ and let V be an open set in Y such that $f(x_0) \in V$. As Y is a metric space there is some $\varepsilon > 0$ such that $B_d(f(x_0), \varepsilon) \subseteq V$. By uniform convergence, there is some $N \in \mathbb{Z}_+$ such that for $n > N$ and all $x \in X$, $d(f_n(x), f(x)) < \frac{\varepsilon}{3}$. Further, as each f_n is continuous at x_0 there is some open set U in X containing x_0 such that $f_n(U) \subseteq B_d(f_n(x_0), \frac{\varepsilon}{3})$. To summarize:

(1) $d(f_n(x_0), f_n(x)) < \frac{\varepsilon}{3}$ for all $x \in U$ and all n, in particular all $n > N$.
(2) $d(f_n(x), f(x)) < \frac{\varepsilon}{3}$ for all $x \in X$ and all $n > N$ by uniform convergence.
(3) $d(f_n(x_0), f(x_0)) < \frac{\varepsilon}{3}$ by uniform convergence applied at x_0.

By repeated application of the Triangle Inequality:

$$\begin{aligned} d(f(x), f(x_0)) &\leq d(f(x), f_n(x)) + d(f_n(x), f(x_0)) \\ &\leq d(f(x), f_n(x)) + d(f_n(x), f_n(x_0)) + d(f_n(x_0), f(x_0)) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon \end{aligned}$$

The above holds for some n large enough and for all fixed x_0, hence for all $x \in X$ there exists an open set U in X such that $x \in U$ and $f(U) \subseteq B_d(f(x), \varepsilon) \subseteq V$. It follows that f is continuous. $\square$

The difference in what we have shown here as compared to in analysis is the restriction to functions of the form $f : \mathbb{R}^n \to \mathbb{R}^m$. In fact, we are not even fixed to Euclidean space at all. The Uniform Limit Theorem will hold for any arbitrary topological space as the domain so long as the codomain is metrizable. This is certainly not a result we have had access to previously. The constant thread is our intuition. We can rely on our understanding of the standard or square metric on $\mathbb{R}^n$ to get a feeling for how open sets in some arbitrary metric space will behave. The more topological properties (first countable, sequentially compact, etc.) that the space has in common with $\mathbb{R}^n$, the better our intuition applies.

10. The Quotient Topology

10.1. **Introduction.** Quotients are the primary way that topologists construct interesting spaces without being tied to ambient space. This is necessary as, for instance, the 'Klein bottle' is a topological space constructed as a quotient which cannot be embedded into $\mathbb{R}^3$ even though at each point there is an open set containing the point that is homeomorphic to an open ball in $\mathbb{R}^2$. While in general every partition of a topological space gives rise to a quotient space, the more common use is to construct spaces from simple, easily understood pieces.

10.2. **Quotient Maps.**

Definition 10.1. Let X and Y be topological spaces. A surjective function $p : X \to Y$ is called a *quotient map* if each subset $V \subseteq Y$ is open if and only if $p^{-1}(U)$ is open in X.

Recall that a function is said to be *open (closed)* if the image of open (closed) sets are open (closed). Quotient maps exhibit a weaker property:

Definition 10.2. A subset $C \subseteq X$ is *saturated* with respect to the surjective function $p : X \to Y$ if $C \cap p^{-1}(\{y\}) \neq \emptyset \Rightarrow p^{-1}(\{y\}) \subseteq C$. That is, there is some set $B \subseteq Y$ such that $C = p^{-1}(B)$.

Proposition 10.3. *The surjective function $p : X \to Y$ is a quotient map if and only if p is continuous and the image of saturated open sets of X are open in Y.*

As a consequence of Proposition 10.3 that a surjective continuous open or closed function is a quotient map. To see the inequivalence, there are quotient maps that are neither open or closed, neither are all open functions closed and vice versa:

Example 10.4. Let $X = [0, 1] \cup [2, 3]$ and $Y = [0, 2]$ define $p : X \to Y$ by

$$p(x) = \begin{cases} x & ; x \in [0, 1] \\ x - 1 & ; x \in [2, 3] \end{cases}$$

We see p, being defined by elementary rules on $[0, 1]$ and $[2, 3]$ is continuous via the pasting lemma. Further p is surjective and closed and is therefore a quotient map. The image of $(\frac{1}{2}, 1]$ which is open in X is $p((\frac{1}{2}, 1]) = (\frac{1}{2}, 1]$ is not open in Y, thus p is not open.

If $A = [0, 1) \cup [2, 3] \subset X$, the restriction $q = p|_A$ is continuous and surjective (Theorem 8.11). However, q is not a quotient map as $[2, 3]$ is open in X and saturated in with respect to q ($[2, 3] = q^{-1}([1, 2])$) but the image $q([2, 3]) = [1, 2]$ is not open in Y.

We have already seen interesting behavior of quotient maps with subspaces. There are certain conditions under which they are better behaved:

Theorem 10.5. *Let $p : X \to Y$ be a quotient map. If A is saturated subset of X with respect to p, then the function $q : A \to p(A)$ obtained by restricting p is such that:*

1. *If A is open or closed in X, then q is a quotient map.*
2. *If p is either open or closed as a function, then q is a quotient map.*

Proof. We prove part (2) here. Let $p : X \to Y$ be an open quotient map and A a saturated subset of X with respect to p. Let $q : A \to p(A)$ be the restriction of p to A. Then q is surjective and continuous. It is left to show that if $q^{-1}(V)$ is open in V then V is open in $p(A)$. To this end, we assume that $q^{-1}(V)$ is open in A. As $V \subseteq p(A)$ and because A is saturated with respect to p, it follows that $p^{-1}(V) \subseteq A$. This inclusion implies that $p^{-1}(V) = q^{-1}(V)$.

Since $p^{-1}(V)$ is open in A, there exists some U open in X such that $p^{-1}(V) = U \cap A$. By surjective we have $p(p^{-1}(V)) = V$ and hence

$$V = p(p^{-1}(V)) = p(U \cap A) = p(U) \cap p(A),$$

where the last equality holds as A is saturated. As p is open, $p(U)$ is open hence $V = p(U) \cap p(A)$ is open in $p(A)$, our desired result. □

Lemma 10.6. *The composition of quotient maps is a quotient map.*

Proof. Let $p : X \to Y$ and $q : Y \to Z$ be quotient maps. The composition $q \circ p$ is the composite of surjective functions and is therefore surjective. Now, suppose W is open in Z. Since q is a quotient map, this is equivalent to $q^{-1}(W)$ is open in Y, which is in turn equivalent to $p^{-1}(q^{-1}(W))$ is open in X. As $(q \circ p)^{-1}(A) = p^{-1}(q^{-1}(A))$ for all subsets $A \subseteq Z$, we have that W is open in Z if and only if $(q \circ p)^{-1}(W)$ is open in X. Thus $q \circ p$ is a quotient map. □

While there is much more to be said about quotient maps in their own right, let us first turn to how they are used to construct new topologies.

Definition 10.7. Let $X, \mathscr{T}_X$ be a topological space and Y some set. If $p : X \to Y$ is surjective, then there exists exactly one topology $\mathscr{T}_Y$ on Y such that p is a quotient map. This topology is the *quotient topology* induced by p. Further,

$$\mathscr{T}_Y = \{V \subseteq Y : p^{-1}(V) \in \mathscr{T}_X\}.$$

The three axioms are easy enough to check to show $\mathscr{T}_Y$ is a topology on Y:

(1) $p^{-1}(Y) = X \in \mathscr{T}_X$ and $p^{-1}(\emptyset) = \emptyset \in \mathscr{T}_X$ implies $Y, \emptyset \in \mathscr{T}_Y$.
(2) If $\{V_1, V_2, \cdots, V_n\} \subset \mathscr{T}_Y$ then

$$\begin{aligned} \forall i \quad p^{-1}(V_i) \in \mathscr{T}_X &\Rightarrow \bigcap_{i=1}^{n} p^{-1}(V_i) \in \mathscr{T}_X \\ &\Rightarrow p^{-1}(\bigcap_{i=1}^{n} V_i) \in \mathscr{T}_X \\ &\Rightarrow \bigcap_{i=1}^{n} V_i \in \mathscr{T}_Y \end{aligned}$$

(3) Similarly, for an arbitrary subcollection $\mathscr{V} = \{V_\alpha\}_{\alpha\in J} \subseteq \mathscr{T}_Y$:

$$\begin{aligned}\forall\alpha \in J \quad p^{-1}(V_\alpha) \in \mathscr{T}_X &\Rightarrow \bigcup_{\alpha\in J} p^{-1}(V_\alpha) \in \mathscr{T}_X \\ &\Rightarrow p^{-1}(\bigcup_{\alpha\in J} V_\alpha) \in \mathscr{T}_X \\ &\Rightarrow \bigcup_{\alpha\in J} V_\alpha \in \mathscr{T}_Y\end{aligned}$$

Uniqueness of the topology $\mathscr{T}_Y$ is due to biconditional in the definition of quotient maps. For if $\mathscr{T}'$ is some other topology on Y with which p is a quotient map and $V \in \mathscr{T}'$, then $p^{-1}(V)$ is open in X implying $V \in \mathscr{T}_Y$. This shows that $\mathscr{T}_Y$ is the finest topology on Y for which p is a quotient map. We find that $\mathscr{T}_Y \subseteq \mathscr{T}'$ holds by a symmetric argument.

Constructing topologies via functions is a general notion. Let X be a topological space and let Y, Z be two sets. Given functions $f : X \to Y$ and $g : Z \to X$, one can consider the finest topology on Y such that f is continuous and likewise the coarsest topology on Z such that g is continuous. In fact, we've seen the latter case explicitly before: the product topology on $X \times Y$ is the coarsest topology on the Cartesian product such that the two projection maps π_1, π_2 are both continuous.

Example 10.8. Let $Y = \{a, b\}$ and define $p : \mathbb{R} \to Y$ by

$$p(x) = \begin{cases} a; & x = 0 \\ b; & x \neq 0 \end{cases}$$

As p is surjective, Definition 10.7 gives the quotient topology on Y as the Sierpinski space

$$\mathscr{T}_Y = \{\emptyset, Y, \{b\}\}.$$

Definition 10.9. Let X be a topological space and X^* a partition of X into disjoint subsets whose union is X. Let $p : X \to X*$ be defined by the rule $p(x) = E$ where $E \in X^*$ is the unique partition such that $x \in E$. As the union of elements in $X*$ is all of X, p is well defined and must be surjective. In the quotient topology induced by p, the space X^* is called the *quotient space* of X.

Recall that every such partition defines an equivalence relation on X and every equivalence relation defines such a partition. Then X^* is the set of equivalence classes and the quotient topology is a topology on the set of equivalence classes of X under some equivalence relation $\sim$. When the equivalence relation is given it is common to write the quotient space as $X^* = X/\!\sim$. The open sets in $X/\!\sim$ are collections $V = \{E_\alpha\}$ of equivalence classes whose union, $\bigcup_\alpha E_\alpha$ is an open set in X.

Example 10.10. Let X be the closed unit ball in $\mathbb{R}^2$:

$$X = \{x \times y \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$$

and let X^* be the partition of X consisting of the one point sets of the form $\{x \times y\}$ whenever $x^2 + y^2 < 1$ and the set $S^1 = \{x \times y \in \mathbb{R} : x^2 + y^2 = 1\}$. The quotient space X^* is homeomorphic to the unit 2-sphere in $\mathbb{R}^3$,

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

The homeomorphism $h : X^* \to S^2$ is given by

$$h(x \times y) = \begin{cases} (x\sin(\pi\sqrt{x^2+y^2}), y\sin(\pi\sqrt{x^2+y^2}), \cos(\pi\sqrt{x^2+y^2})) & \text{if } x^2+y^2 < 1 \\ (0,0,-1) & \text{if } x \times y \in S^1 \end{cases}$$

Example 10.11. Let X be the square $[0,1] \times [0,1]$. Define a equivalence relation $\sim_S$ by $x \times 0 \sim_S 0 \times x$ and $x \times 1 \sim_S 1 \times x$ and point-wise equality elsewhere on X. The quotient space $X/\sim_S$ is homeomorphic to S^2 as shown in the homework.

The equivalence relation $\sim_K$ given by $x \times 0 \sim_K 1 \times x$ and $x \times 1 \sim_K 0 \times x$, yields a quotient space $K = X/\sim_K$ known as the Klein bottle. What is not clear at our current vantage is if K is homeomorphic to $X/\sim_S$ or some other possible analogous quotient spaces on $[0,1] \times [0,1]$.

Before returning to the topic of functions and quotient spaces, we make a remark about Hausdorff spaces.

Remark 10.12. The quotient of a Hausdorff space is not necessarily Hausdorff. Consider the space $\mathbb{R}_K$ and let Y be the quotient space obtained by identifying the set K to a point. Let $p : \mathbb{R}_K \to Y$ denote the associated quotient map. Let $V \subseteq Y$ be an open set containing the equivalence class K. By definition, $p^{-1}(V)$ is open in $\mathbb{R}_K$ and $K \subset p^{-1}(V)$. Let $U \subseteq Y$ be an open set containing the equivalence class $\{0\}$. Then $0 \in p^{-1}(U)$ is open in $\mathbb{R}_K$. We will show $p^{-1}(U) \cap p^{-1}(V) \neq \emptyset$ implying $U \cap V \supseteq p(p^{-1}(U) \cap p^{-1}(V))$ is non-empty.

If $p^{-1}(U) \cap K \neq \emptyset$ we are done. Assume $p^{-1}(U) \cap K = \emptyset$. As $p^{-1}(U)$ is open there must be some basic open set $B = (a,b) - K$ such that $a < 0 < b$ and $B \subseteq p^{-1}(U)$. Note there exists some $N \in \mathbb{Z}_+$ such that $\frac{1}{N} \in (a,b)$. Returning to $p^{-1}(V)$, $\frac{1}{N} \in p^{-1}(V)$ with openness implying there is a basic open set $C = (c,d)$ containing $\frac{1}{N}$ with $C \subseteq p^{-1}(V)$. Then $C \cap B = (\max\{a,c\}, \min\{b,d\}) - K$ is non-empty. Concluding, we have $p(C \cap B) \subseteq p(p^{-1}(U) \cap p^{-1}(V)) \subseteq U \cap V$ is non-empty.

We have shown that for all open sets containing the class K and 0 is Y respectively, they must have non-empty intersection. Thus Y is not Hausdorff.

The solution to the problem of constructing continuous functions whose domain is a quotient space takes the form of a universal property.

Theorem 10.13. *Let X, Y and Z be topological spaces and let $p : X \to Y$ be a quotient map. Suppose $g : X \to Z$ is a function which is constant on the subsets $p^{-1}(\{y\})$ for each $y \in Y$. Then g induces a function $f : Y \to Z$ such that the following diagram commutes:*

$$\begin{array}{ccc} X & & \\ \downarrow{\scriptstyle p} & \searrow{\scriptstyle g} & \\ Y & \overset{f}{\dashrightarrow} & Z \end{array}$$

The induced function f is continuous if and only if g is continuous; f is a quotient map if and only if g is a quotient map.

Proof. Let X, Y, Z, p, g be given as in the theorem statement. We note that as p is surjective, each set $p^{-1}(\{y\})$ is nonempty and $g(p^{-1}(\{y\}))$ is a singleton in Z. We define $f(y)$ to be this point. That is, $f : Y \to Z$ such that $f(p(x)) = g(x)$ for all $x \in X$.

Suppose f is continuous. Then g is the composition of continuous functions and is hence continuous. To the converse, suppose g is continuous and let W be open in Z. By continuity, $g^{-1}(W)$ is open in X. As $g^{-1}(W) = p^{-1}(f^{-1}(W))$, the fact that p is a quotient map implies $f^{-1}(W)$ is open in Y. Thus f is continuous.

Supposing f is a quotient map, by Lemma 10.6 g is also a quotient map. Conversely, suppose that g is a quotient map and assume $f^{-1}(W)$ is open in Y for some $W \subseteq Z$. By continuity of p, $p^{-1}(f^{-1}(W))$ is open in X. Rewriting $p^{-1}(f^{-1}(W)) = g^{-1}(W)$, we have that W is open in Z as g is a quotient map. Thus f is a quotient map, having the property that W is open in Z if and only if $f^{-1}(W)$ is open in Y. □

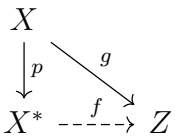
As a direct consequence, there is an typical way to construct homeomorphisms between quotient spaces:

Corollary 10.14. *Let $g : X \to Z$ be a surjective continuous map. Let X^* be the partitions of X given by*

$$X^* = \{g^{-1}(\{z\}) : z \in Z\}.$$

Give X^* the quotient topology induced by the quotient map $p : X \to X^*$.

(1) *g induces a bijective continuous function $f : X^* \to Z$ such that the following diagram commutes.*



The function f is a homeomorphism if and only if g is a quotient map.

(2) *If Z is Hausdorff, so is X^*.*

11. Connectedness

11.1. **Introduction.** We are now to the point in our studies that we can introduce (rapidly I may add) new topological properties. The first will be connectedness and those related ideas. It can be said that such topological properties is key to determining the homeomorphism class of spaces. Up to this point, determining if two spaces are homeomorphic can be quite challenging. However, if we can determine that one space is, say, connected while the other is not, our job will be finished!

11.2. **Connected Spaces.**

Definition 11.1. Let X be a topological space. If there exists open sets U, V in X such that $U \cup V = X$ and $U \cap V = \emptyset$ then X is said to be *disconnected*. A space that is not disconnected is called *connected*.

Munkres terms the sets U and V as a *separation* of X. We want to be careful with the terminology as a space being "separable" is entirely unrelated to separations. Though, giving a name to this construct can be helpful. Notice that $X - U = V$ and $X - V = U$ are both open, hence both U and V are closed. Therefore a separation of X exists if and only if there is a proper, non-empty subset U that is both open and closed in X. As mentioned in the section on closed sets, we sometimes refer to such a set as *clopen*.

Lemma 11.2. *A space X is connected if and only if the only clopen sets are X and $\emptyset$.*

There is a convenient way to phrase clopen sets in terms of limit points.

Lemma 11.3. *Let X be a topological space with non-empty subset U. Then U is clopen if and only if neither set U or $V = X - U$ contain a limit point of the other.*

Proof. Let X, U, V be given as in the statement of the lemma. Note that $U \cap V = \emptyset$. Suppose U is clopen. Then it is by definition closed, and hence contains its own limit points. As the intersection with V is empty, V contains no limit point of U. Further, as U is open V is closed and the same argument implies that U contains no limit point of V.

Suppose that neither U or V contain a limit point of the other set. The closure of U is the union of U with its set of limit points U'. It was assumed that $U' \cap V = \emptyset$ and as $X - V = U$, we have $U' \subseteq U$. Thus U is closed. By the same argument V is closed. This again implies U and V are both open. In particular, U is clopen. □

The reader is asked to determine the implication of the above on the related lemma as quoted from Munkres:

Lemma 11.4 (Lemma 23.1 in Munkres)**.** *If Y is a subspace of X, a separation of Y is a pair of disjoint nonempty subsets A and B whose union is Y, neither of which contains a limit point of the other. The space Y is connected if there exists no separation of Y.*

Example 11.5. Every finite set with the indiscrete topology is connected: the only clopen sets are the whole set and $\emptyset$.

Every set with more than one element equipped with the discrete topology is disconnected: every set is both open and closed.

Example 11.6. The rationals $\mathbb{Q}$ equipped with the subspace topology from $\mathbb{R}$ is disconnected. In fact every subset Y of $\mathbb{Q}$ is disconnected. If $p, q \in Y$ with $p < q$ then choose an irrational number x such that $p < x < q$. $Y \cap (-\infty, x)$ is open being the intersection of Y with an open set and is closed since $Y \cap (-\infty, x] = Y \cap (-\infty, x)$.

Example 11.7. Constructing disconnected subspaces of $\mathbb{R}$ is generally straightforward, simply make the whole set as a separation. $Y = [0, 1] \cup (2, 3]$ is disconnected.

Let us focus on how we can construct connected spaces.

Lemma 11.8. *Let X be a disconnected space and let U be a clopen set. If Y is a connected subspace of X then either $Y \subseteq U$ or $Y \subseteq X - U$.*

Proof. We shall prove the contrapositive: suppose that Y is not contained in either U or $X - U$. As $U \cup (X - U) = X$, $Y \cap U$ is non-empty nor does it equal all of Y. As U is clopen $Y \cap U$ is clopen in Y. Thus Y is disconnected. □

There is a natural construction to break up a disconnected space into the "largest" connected subspaces.

Definition 11.9. Given a space X, define an equivalence relation on X by setting $x \sim y$ if there is a connected subspace of X containing both x and y. The equivalence classes are called *connected components* or just components for brevity.

So a space is disconnected if and only if it has more than one connected component. Though defined as equivalence classes, we use the connected components as subsets of X. This is as opposed to points in a quotient space.

Theorem 11.10. *The union of connected subspaces of X that have a point in common is itself connected.*

Proof. Let $\{A_\alpha\}$ be the collection of connected subspaces of X and let $a \in \bigcap_\alpha A_\alpha$. We will show that the only non-empty clopen subset of $Y = \bigcup_\alpha A_\alpha$ is Y itself. Assume U is a clopen in Y. Without loss of generality, assume that $a \in U$ (otherwise proceed with $Y - U$). As $a \in A_\alpha$ and A_α is connected, by Lemma 11.8 $A_\alpha \subseteq U$. This is true for all α, thus $Y = \bigcup_\alpha A_\alpha \subseteq U$. As $U \subseteq Y$ we have the desired equality. By Lemma 11.2 Y, the union of connected subspaces, is connected. □

We have a result similar to Lemma 11.8 regarding connected components:

Theorem 11.11. *The connected components of X are connected disjoint subsets of X whose union is all of X. Any connected subspace of X must be contained entirely within some connected component.*

Proof. As the components are equivalence classes they form a partition of X. For A a connected subspace of X and for connected components C_1, C_2, if $A \cap C_1$ and $A \cap C_2$ are non-empty we can choose $x_1 \in A \cap C_1$ and $x_2 \in A \cap C_2$. Then $x_1 \sim x_2$ as A contains both x_1 and x_2, hence they belong to the same equivalence class implying $C_1 = C_2$. Further all points in A belong to the same equivalence class, and we conclude $A \subseteq C_1 = C_2$.

To show the components are themselves connected, choose a point c_0 is some component C. For each point $x \in C$ we know $x \sim c_0$, that is there is a connected

subspace A_x such that $x, c_0 \in A_x$. By the above, we have $A_x \subseteq C$. Then

$$\bigcup_{x \in C} A_x = C$$

and as each A_x has c_0 in common, C is connected by Theorem 11.10. □

Lemma 11.12. *Let A be a connected subspace of X. If $A \subseteq B \subseteq \bar{A}$, then B is connected.*

Proof. Let A be a connected subset of X and suppose B is such that $A \subseteq B \subseteq \bar{A}$. We will show that the only non-empty set that is both open and closed in B is B itself. Suppose C is a non-empty clopen subset in B. By Lemma 11.8, A is contained in either C or $B - C$. Without loss of generality we assume $A \subseteq C$. Then $\bar{A} \subseteq \bar{C}$ where the closure is taken in X. We then have

$$B = B \cap \bar{A} \subseteq B \cap \bar{C} = C$$

where the last equality holds as C is closed in B. Thus $B = C$ and the only non-empty clopen subset of B is B. By Lemma 11.2 B is connected. □

Remark 11.13. One useful consequence of Lemma 11.12 is the closure of a connected subset is connected. (Note this result is not mirrored for the interior of a set:

$$X = \{x \times y : x \geq 0 \wedge y \geq 0\} \cup \{x \times y : x \leq 0 \wedge y \leq 0\}$$

is a connected subspace of $\mathbb{R}^2$ whose interior is disconnected.) What this result says about connected components is they must be closed. Further if there are finitely many connected components then each is also open. This last statement is not true in general: in $\mathbb{Q}$ the connected components are one-point sets which are not open in $\mathbb{Q}$.

We now see a direct application of continuous functions to the exploration of topological properties and perhaps further justify our definition of continuity.

Lemma 11.14. *Let $f : X \to Y$ be a continuous function. If X is connected, then $f(X)$ is connected.*

Proof. We need only consider the case $f(X) = Y$ as for any codomain restriction $f(X) \subseteq Z \subseteq Y$ the map $f : X \to Z$ will have the same conclusion. We proceed by way of contradiction. Suppose the image is disconnected, that is, there exists some non-empty clopen proper subset U of Y. Surjectivity of f onto Y implies that $f^{-1}(U)$ is a proper non-empty subset of X. As f is assumed to be continuous and U is open, $f^{-1}(U)$ is open. As U is also closed, by Theorem 8.6, $f^{-1}(U)$ is closed. Hence $f^{-1}(U)$ is a proper non-empty subset of X that is clopen, contradicting the assumption that X is connected. Therefore, $f(X)$ must be connected. □

Theorem 11.15 (The Intermediate Value Theorem)**.** *Let X be a connected space and let Y be a set with a strict total order endowed with the order topology. Let $f : X \to Y$ be continuous. If $a, b \in X$ and there exists $r \in Y$ such that $f(a) < r < f(b)$, then there exists some $c \in X$ such that $f(c) = r$.*

Proof. Let X, Y, f, a, b, r be given as in the statement of the theorem and assume that r is not in the image of X under f. Then $A = f(X) \cap (-\infty, r)$ and $B = f(X) \cap (r \infty)$ are open in the subspace topology on $f(X)$ and are such that $A \cup B = X$. Then

$f(X)$ is disconnected, contradicting Lemma 11.14. Thus r is in the image of X, that is to say there exists some $c \in X$ such that $f(c) = r$. □

Theorem 11.16. *A finite Cartesian product of connected spaces is connected.*

Proof. Let X and Y be connected spaces. Choose a point $a \times b \in X \times Y$. The subspace $X \times \{b\}$ is connected as it is the image of the continuous function $f : X \to X \times Y$ given by $f(x) = x \times b$. By the same argument, the subspace $\{x\} \times Y$ is connected for all $x \in X$. By Theorem 11.10, the subspace $T_x = (X \times \{b\}) \cup (\{x\} \times Y)$ is connected being the union of two connected subspaces with the point $x \times b$ is common. We extend this further to $\bigcup_{x \in X} T_x$ which is again connected by Theorem 11.10 being the union of connected subspaces having the point $a \times b$ is common. Noting finally that $\bigcup_{x \in X} T_x = X \times Y$ we conclude that $X \times Y$ is connected.

Connectedness for finite Cartesian products $X_1 \times X_2 \times \cdots \times X_n$ follows inductively by setting $X = X_1 \times \cdots \times X_{n-1}$ and $Y = X_n$. □

We now want to find a familiar connected space.

Lemma 11.17. *The interval* $[0, 1]$ *in* $\mathbb{R}$ *is connected.*

Proof. Suppose $[0, 1]$ was disconnected. Then there must exist disjoint nonempty sets U and V such that both U and V are open and $U \cup V = [0, 1]$. As U and V are disjoint it must be that one does not contain the real number 1, assume $1 \notin U$ (equivalently $1 \in V$).

Let $c = \sup U$. We will show that c does not belong to either U nor V contradicting that $[0, 1]$ is the union of U and V. As U is non-empty and open it contains some basic open set of the form (a, b) or $[0, d)$. In particular $c > 0$.

Suppose $c \in V$. Either $c = 1$ or $0 < c < 1$. In both cases, openness of V implies that there is some interval of the form $(d, c] \subseteq V$. We can extend this to the interval $(d, 1] \subseteq V$ or otherwise there would be some $c' \in U$ such that $c' > c$. Even still, $d < c$ and have d is an upper bound of U by the previous sentence. Again contradicting that c is the supremum of U.

Now we suppose that $c \in U$. Then $c \neq 1$ and by openness of U there is some interval $[c, d) \subseteq U$. By completeness of the real numbers, there is some number e such that $c < e < d$. It follows that $e \in U$ contradicting that c is the supremum of U.

We conclude that c is in neither U nor V and therefore $U \cup V \neq [0, 1]$. Thus $[0, 1]$ must be connected. □

A similar argument can be made to show that other intervals of $\mathbb{R}$ are connected, but more interesting is the introduction of the following topological property.

11.3. **Path Connected Space.**

Definition 11.18. Given points $x, y \in X$, a *path* in X from x to y is a continuous function $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$. A space is *path connected* if every pair of points of X can be joined by a path in X.

Lemma 11.19. *If a space X is path connected, then X is connected.*

Proof. We utilize a proof by contradiction. Assume X is path connected and not connected. Then there exists some non-empty proper clopen subset U of X. Let $x \in U$ and $y \in X - U$. By path connectedness of X there exists a continuous function $f : [0, 1] \to X$ from x to y. By Lemma 11.14 the image of $[0, 1]$ is connected. Lemma 11.8 then implies that the image is entirely contained in either U or $X - U$, contradicting the fact that $f(0) = x$ and $f(1) = y$. □

Corollary 11.20. *The real line $\mathbb{R}$ is connected as is every interval and ray in $\mathbb{R}$.*

Proof. For all the above sets, if a pair x, y is in the set then the interval $[x, y]$ is a subset. The function $f : [0, 1] \to \mathbb{R}$ given by $f(t) = t(y - x) + x$ yields the result that the space is path connected and hence connected. □

Example 11.21. Euclidean space $\mathbb{R}^n$ is connected as is every open ball $B_d(\mathbf{x}, \varepsilon)$. For any two points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ the straight line $f : [0, 1] \to \mathbb{R}^n$ given by $f(t) = t(\mathbf{y} - \mathbf{x}) + \mathbf{x}$ yields path connectedness. Further if $\mathbf{x}, \mathbf{y} \in B_d(\mathbf{z}, \varepsilon)$ then $f([0, 1]) \subset B_d(\mathbf{z}, \varepsilon)$.

We can achieve the same result by Corollary 11.20 and Theorem 11.16 that $\mathbb{R}^n$ is connected. The homeomorphism $h : \mathbb{R}^n \to B_d(\mathbf{x}, \varepsilon)$ given by $h(\mathbf{y}) = \mathbf{x} + \frac{\varepsilon}{1+||\mathbf{y}||}\mathbf{y}$ concludes the claim.

Example 11.22. Punctured Euclidean space $\mathbb{R}^n - \{\mathbf{0}\}$ is path connected and hence connected. Given any two points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n - \{\mathbf{0}\}$ the straight line $f : [0, 1] \to \mathbb{R}^n$ given by $f(t) = t(\mathbf{y} - \mathbf{x}) + \mathbf{x}$ yields path connectedness so long as it does not go through the origin. Otherwise let z be a point not in the image of f and define the path $g : [0, 1] \to \mathbb{R}^n - \{\mathbf{0}\}$ by

$$g(x) = \begin{cases} 2t(\mathbf{z} - \mathbf{x}) + \mathbf{x} & \text{for } 0 \leq t \leq \frac{1}{2} \\ 2(t-1)(\mathbf{y} - \mathbf{z}) + \mathbf{y} & \text{for } \frac{1}{2} < t \leq 1 \end{cases}$$

Example 11.23. The unit sphere S^n in $\mathbb{R}^{n+1}$ given by

$$S^n = \{\mathbf{x} \in \mathbb{R}^{n+1} : ||\mathbf{x}|| = 1\}$$

is continuous at it is the image of a connected space under a continuous function. Namely for the map $f : \mathbb{R}^{n+1} - \{\mathbf{0}\} \to S^n$ given by $g(\mathbf{x}) = \frac{\mathbf{x}}{||\mathbf{x}||}$.

Example 11.24. The space $S = \{x \times \sin(\frac{1}{x}) : x > 0\}$ is the image of the connected set $(0, \infty)$ under a continuous function and is thus continuous. By Lemma 11.12, $\bar{S}$ is also connected. This space is referred to as the "topologist's sine curve" though some sources will restrict $0 < x \leq 1$. Despite being connected, $\bar{S}$ is not path connected.

Let $x \in \{0\} \times [-1, 1]$ and $y \in S$ and suppose $f : [0, 1] \to \bar{S}$ was a path from x to y. The set $\{0\} \times [-1, 1]$ is closed hence $f^{-1}(\{0\} \times [-1, 1])$ is closed. Thus it is abounded closed set and contains its largest element, say b. Let $f' : [0, 1] \to \bar{S}$ be the linear

reparametrization of $f|_{[b,1]}$ given by $f'(t) = f((1-b)t+b)$. Then f' is a path from $f(b)$ to y such that $f'(0) = f(b)$ and $f'(t) \in S$ for all $t > 0$.

Write $f'(t) = (x(t), y(t))$ and we have $x(0) = 0$ and $x(t) > 0, y(t) = \sin\frac{1}{x(t)}$ for all $t > 0$. Given $n \in \mathbb{Z}_+$ choose some $u \in (0, x(1/n))$ such that $\sin(\frac{1}{u}) = (-1)^n$. By the Intermediate Value Theorem applied to $x(t)$ we can find some $t_n \in (0, \frac{1}{n})$ such that $x(t_n) = u$.

We now have $y(t_n) = (-1)^n$ and by the contrapositive of Theorem 9.15 $y(t)$ is not continuous. Thus by the contrapositive of Theorem 8.14 f' is not continuous, a contradiction. We conclude that no such path f can exist, hence $\bar{S}$ is not path connected.

Similar to the connected components we define the following for path connectedness:

Definition 11.25. Let X be a space. We define an equivalence relation on X by $x \sim y$ if there exists a path in X from x to y. The equivalence classes are the *path components*.

Analogous to Theorem 11.11 we have for path components:

Theorem 11.26. *The path components of X are path connected disjoint subspaces of X whose union is X. Each non-empty path connected subspace of X is contained entirely in a single path component*

Example 11.27. Path components are certainly distinct from connected components. The topologist's sine curve is connected and hence has only one connected component, but is not path connected so at least has two path components. In fact, the space has exactly two path components: $\{0\} \times [-1, 1]$ and $\{x \times \sin(\frac{1}{x}) : x > 0\}$.

11.4. **Local Connectedness.** We now briefly look at some local topological properties, those that only occur at certain points.

Definition 11.28. A space X is *locally connected at x* if for every open set containing x there is a connected open set V containing x such that $V \subseteq U$. If X is locally connected for all $x \in X$ then the space X is said to be *locally connected.* Similarly, a space X is *locally path connected at x* if for every open set U containing x there is a path connected open set V containing x such that $V \subseteq U$. If X is locally path connected for all $x \in X$ then X is said to be *locally path connected.*

Example 11.29. These local properties are distinct from their global counterparts. The subspace $[0, 1) \cup (1, 2]$ in $\mathbb{R}$ is disconnected but locally connected. The topologist's sine curve is connected by not locally connected, it fails to be locally connected at each point in $\{0\} \times [-1, 1]$.

The connection with the global property is illustrated, shoring up Remark 11.13:

Theorem 11.30. *A space X is locally (path) connected if and only if for every open set U of X, each (path) component of U is open in X.*

Theorem 11.31. *For a space X, each path component of X lies in a component of X. If X is locally path connected, then the components and the path components are the same.*

Proof. Let C be a component of X and let $x \in C$. Suppose P is the path component of X containing x. Since P is connected, $P \subseteq C$; the first claim is proven.

Now suppose X is locally path connected. By way of contradiction, suppose P is a proper subset of C. Let Q be the union of all path components of X which have non-empty intersection with C and are disjoint from P. As each path component is connected, $Q \subset C$ and we further have that $C = P \cup Q$. By local path connectedness of X, each of P and Q are open. As $C - P = Q$, we have that P is closed. By Lemma 11.2, C is disconnected contradicting the conclusion of Theorem 11.11. □

We immediately have:

Corollary 11.32. *If X is connected and locally path connected then X is path connected.*

12. Compactness

12.1. **Introduction.** One may recall the beginning of seemingly every theorem from calculus, "let f be a real valued function continuous on the closed interval $[a, b]$. There is clearly something special about such a domain. In this section we look at that special property and generalize it to arbitrary spaces.

12.2. **Compact Spaces.**

Definition 12.1. A collection $\mathcal{A}$ of subsets of a space X such that $X = \bigcup_{A\in\mathcal{A}} A$ is said to *cover* X and is called a *covering of* X. If each $A \in \mathcal{A}$ is open in X, then $\mathcal{A}$ is called an *open covering of* X.

Definition 12.2. A space X is *compact* if every open covering of X contains a finite subcollection covering X.

Example 12.3. The real line $\mathbb{R}$ is not compact. For the open covering

$$\mathcal{A} = \{(n, n + 2) : n \in \mathbb{Z}\}$$

covers $\mathbb{R}$ but for each $i \in \mathbb{Z}$, the only interval containing i is $(i - 1, i + 1)$. Thus a finite subcollection of $\mathcal{A}$ must miss some integer (infinitely many, in fact) and thus is not a covering of $\mathbb{R}$.

Example 12.4. The interval $(0, 1]$ is not compact; consider the open covering

$$\mathcal{A} = \{(1/n, 1] : n \in \mathbb{Z}_+\}.$$

If $\mathcal{A}_0$ is a finite subcollection of $\mathcal{A}$, then there is a maximal integer n such that $(1/n, 1] \in \mathcal{A}_0$. Then $\frac{1}{n+1} \notin \bigcup_{A\in\mathcal{A}_0} A$ and we conclude that A_0 is not a covering of $(0, 1]$.

So, what spaces are compact? It is, in general, not so easy to show a space is compact. It does not seem at first straightforward to check that EVERY covering has a finite subcollection which is also a covering. Let us start developing our toolbox with subspaces:

Definition 12.5. For a subspace Y of X, a collection $\mathcal{A}$ of subsets of X such that $Y \subseteq \bigcup_{A\in\mathcal{A}} A$ is said to *cover* Y.

Lemma 12.6. *Let X be a topological space and Y a subspace of X. Y is compact if and only if every covering of Y by open subsets of X contains a finite subcollection covering Y.*

Proof. Let Y be a subspace of X.

($\Rightarrow$) Suppose Y is compact and let $\mathcal{A}$ be a collection of open subsets of X covering of Y. The collection

$$\mathcal{B} = \{A \cap Y : A \in \mathcal{A}\}$$

covers Y and as each A is open in X, $A \cap Y$ is open in the subspace Y. As Y is compact and $\mathcal{B}$ is an open covering of Y, there is a finite subcollection

$$\{A_1 \cap Y, A_2 \cap Y, \ldots, A_n \cap Y\} \subseteq \mathcal{B}$$

that covers Y. Then

$$\{A_1, \ldots, A_n\}$$

is a finite subcollection of $\mathcal{A}$ which covers Y.

($\Leftarrow$) Suppose Y has the property that for every collection of open subsets of X covering Y, there is a finite subcollection covering Y. To show Y is indeed compact, let $\mathcal{A}'$ be an open covering of Y. That is, each $A' \in \mathcal{A}'$ is open in the subspace Y. By the definition of the subspace topology, for each $A' \in \mathcal{A}'$ we can choose some A open in X such that $A' = A \cap Y$. Define $\mathcal{A}$ to be the collection of open sets in X chosen in this way for all elements of $\mathcal{A}'$. Hence $\mathcal{A}$ is a collection of open subsets of X which covers Y. By our hypothesis, there exists a finite subcollection

$$\{A_1, \ldots, A_n\} \subseteq \mathcal{A}$$

which covers Y. Then

$$\{A_1 \cap Y, \ldots, A_n \cap Y\}$$

is a finite subcollection of $\mathcal{A}'$ which covers Y. Thus Y is compact. □

Let's look at the interplay of compact spaces with some of our previous concepts.

Theorem 12.7. *Every closed subspace of a compact space is compact.*

Proof. Let Y be a closed subspace of a compact space X. By Lemma 12.6 we need only consider coverings of Y by collections of open subsets of X in order to show Y is compact. To that end, let $\mathcal{A}$ be a collection of open subsets of X which cover Y. Define

$$\mathcal{B} = \mathcal{A} \cup \{X - Y\}.$$

As $\mathcal{B}$ is an open cover of X, a compact space, there exists a finite subcollection $\mathcal{B}'$ which covers X. Then

$$\mathcal{A}' = \mathcal{B}' - \{X - Y\}$$

is a finite subcollection of $\mathcal{A}$ that covers Y. Thus by Lemma 12.6, Y is compact. □

Theorem 12.8. *Every compact subspace of a Hausdorff space is closed.*

Proof. Let Y be a compact subspace of a Hausdorff space X. We will show that $X - Y$ is open.

Fix $x \in X - Y$. As X is Hausdorff, for every $y \in Y$ there exists open sets U_y, V_y such that $x \in U_y, y \in V_y$ and $U_y \cap V_y = \emptyset$. The collection $\{V_y : y \in Y\}$ is an covering of Y be open sets of X. As Y is compact there is a finite subcollection

$$\{V_{y_1}, V_{y_1}, \ldots, V_{y_n}\}$$

that covers Y. That is, $V = \bigcup_{i=1}^{n} V_{y_i}$ contains Y. Thus $U = \bigcap_{i=1}^{n} U_{y_i}$ is the finite intersection of open sets and thus open with the additional property that $U \cap V = \emptyset$. Hence $U \subseteq X - Y$.

As $x \in X - Y$ was arbitrary, we conclude for all $x \in X - Y$ there exists an open set U such that $x \in U \subseteq X - Y$. Hence $X - Y$ is open and we conclude that Y is closed. □

We have several consequences of the preceding result:

Lemma 12.9. *If X is a Hausdorff space, then for any compact subspace Y and any $x \notin Y$ there exists open sets U, V such that $x \in U, Y \subseteq V$ and $U \cap V = \emptyset$.*

Corollary 12.10. *Let X be a compact Hausdorff space. Then the collection of closed subspaces and the collection of compact subspaces are equal as sets.*

Example 12.11. As $\mathbb{R}$ is Hausdorff, all intervals of the forms (a, b); $[a, b)$; and $(a, b]$ are not compact.

As we hope, continuous functions preserve compactness.

Theorem 12.12. *Let $f : X \to Y$ be a continuous function with X a compact space. Then $f(X)$ is compact.*

Proof. Let f, X be as in the antecedent. Let $\mathcal{A}$ be a collection of open sets in Y that cover $f(X)$. That is $f(X) \subseteq \bigcup_{A \in \mathcal{A}} A$. As f is continuous and each $A \in \mathcal{A}$ is open, $f^{-1}(A)$ is open in X. We then have

$$X = \bigcup_{A \in \mathcal{A}} f^{-1}(A).$$

Then $\{f^{-1}(A) : A \in \mathcal{A}\}$ is an open covering of X and by compactness there exists a finite subcollection $\{f^{-1}(A_i) : A_i \in \mathcal{A}\}_{i \leq n}$ that covers X. Thus

$$\begin{aligned} f(X) &= f(\bigcup_{i=1}^{n} f^{-1}(A_i)) \\ &= \bigcup_{i=1}^{n} f(f^{-1}(A_i)) && \text{(functions preserve unions)} \\ &\subseteq \bigcup_{i=1}^{n} A_i. && (f(f^{-1}(A_i)) \subseteq A_i) \end{aligned}$$

We conclude that $\{A_1, \ldots, A_n\}$ is a finite subcollection of $\mathcal{A}$ covering $f(X)$. By Lemma 12.6, $f(X)$ is compact.

□

We have some help constructing homeomorphisms from compact spaces to Hausdorff spaces:

Theorem 12.13. *Let $f : X \to Y$ be a bijective continuous function. If X is compact and Y is Hausdorff, then f is a homeomorphism.*

Proof. It suffices to show that f is a closed function. Let $C \subseteq X$ be closed. As X is compact, by Theorem 12.7 we have C is compact. By Theorem 12.12, $f(C)$ is compact. As Y is Hausdorff, Theorem 12.8 $f(C)$ is closed. Thus f is a closed, bijective, continuous function and hence a homeomorphism from X to Y. □

In order to prove the product of finitely many compact spaces is compact we need the following resource:

Lemma 12.14 (The Tube Lemma)**.** *Let X be some topological space and Y a compact space. Consider the product space $X \times Y$. If N is an open set of $X \times Y$ containing the set $\{x_0\} \times Y$ for some $x_0 \in X$, then there is an open set W of X such that $x_0 \in W$ and $W \times Y \subseteq N$.*

Proof. Suppose X, Y are as specified. Suppose N is open in $X \times Y$ and $\{x_0\} \times Y \subseteq N$ for some $x_0 \in X$.

Since N is open, for every point $x_0 \times Y \in \{x_0\} \times Y$ there is a basic open set $U \times V$ where U is open in X, V is open in Y, and $x_0 \times y \in U_y \times V_y \subseteq N$. The collection $\{V_y : y \in Y\}$ is an open cover of Y; as Y is compact, there is a finite subcollection $\{V_{y_i}\}_{i \leq n}$ which covers Y. Thus $\{U_{y_i} \times V_{y_i}\}_{i \leq n}$ covers $\{x_0\} \times Y$. Combining the fact that $x_0 \in W = \bigcap_{i=1}^{n} U_{y_i} \subseteq U_{y_i}$ for all $i \leq n$ with $\{x_0\} \times Y \subseteq \bigcup_{i=1}^{n} \left(U_{y_i} \times V_{y_i}\right) \subseteq N$, we conclude that W is the desired open set of X such that $x_0 \in W$ and $\{x_0\} \times Y \subseteq W \times Y \subseteq N$. □

The Tube Lemma truly lives up to its name when X is a metric space:

Corollary 12.15. *Let (X, d) be a metric space and Y a compact space. Let N be an open set in the product space $X \times Y$ such that the set $\{x_0\} \times Y \subseteq N$. Then there exists some $\varepsilon > 0$ such that the tube $B_d(x_0, \varepsilon) \times Y$ is contained in N.*

We can now return to the matter of the compactness of product spaces.

Theorem 12.16. *The product of finitely many compact spaces is compact.*

Proof. Let X and Y be compact spaces and let $\mathcal{A}$ be an open covering of $X \times Y$. For every $x \in X$ we have $\{x\} \times Y$ is compact hence there is a finite subcollection $\mathcal{A}_x$ which covers $\{x\} \times Y$. As $\bigcup_{A \in \mathcal{A}_x} A$ is open and contains $\{x\} \times Y$, by the Tube Lemma there exists an open set W_x such that $x \in W_x$ and $W_x \times Y$ is covered by $\mathcal{A}_x$. The collection $\{W_x : x \in X\}$ is an open covering of X and hence there is a finite subcollection $W_{x_1}, \ldots, W_{x_n}$ which covers X. As each W_{x_i} is covered by a finite subcollection $\mathcal{A}_{x_i}$ of $\mathcal{A}$ and $\bigcup_{i=1}^{n} W_{x_i} \times Y = X \times Y$, we conclude that $\bigcup_{i=1}^{n} \mathcal{A}_{x_i}$ is a finite (finite union of finite sets) subcollection of $\mathcal{A}$ that covers $X \times Y$. Therefore, as $\mathcal{A}$ is an arbitrary open cover, $X \times Y$ is compact.

The proof that the product $X_1 \times X_2 \times \cdots \times X_n$ of compact spaces is compact follows inductively by setting $X = X_1 \times X_2 \times \cdots \times X_{n-1}$ and $Y = X_n$. □

We may also formulate compactness in terms of closed sets as follows:

Definition 12.17. A collection $\mathcal{C}$ of subsets of X is said to have the *finite intersection property* if for every finite subcollection

$$\{C_1, C_2, \ldots, C_n\}$$

we have $\bigcap_{i=1}^{n} C_i \neq \emptyset$.

Theorem 12.18. *Let X be a topological space. Then X is compact if and only if every collection $\mathcal{C}$ of closed sets having the finite intersection property has the additional property that $\bigcap_{C\in\mathcal{C}} C \neq \emptyset$.*

Proof. By definition the space X is compact if and only if for every open covering $\mathcal{A}$ there is a finite subcollection covering X. Let $\mathcal{C}$ be a collection of closed sets in X. The collection $\mathcal{C}$ has the property that $\bigcap\{C \in \mathcal{C}\}C = \emptyset$ if and only if $\mathcal{A} = \{X - C : C \in \mathcal{C}\}$ is an open cover of X. There exists a finite subcollection $\{C_1, C_2, \ldots, C_n\}$ of $\mathcal{C}$ such that $\bigcap_{i=1}^{n} C_i = \emptyset$ if and only if the subcollection $\{X - C_1, \ldots, X - C_n\}$ of $\mathcal{A}$ is a cover of X.

The necessary and sufficient condition of compactness can then be restated: if $\mathcal{C}$ is a collection of closed sets in X having the property that $\bigcap_{C\in\mathcal{C}} = \emptyset$, then there exists a finite subcollection $\{C_1, C_2, \ldots, C_n\}$ of $\mathcal{C}$ such that $\bigcap_{i=1}^{n} C_i = \emptyset$.

Taking the contrapositive of this conditional we arrive at the conclusion of the theorem: X is compact if and only if every collection $\mathcal{C}$ of closed sets having the finite intersection property has the additional property that $\bigcap_{C\in\mathcal{C}} C \neq \emptyset$. □

12.3. **Compact Subspaces of $\mathbb{R}^n$.** We are now prepared to identify some compact spaces.

Theorem 12.19. *Let X be a set with a strict total order having the least upper bound property. When X is endowed with the order topology, every closed interval is compact.*

Proof. Let $a < b$ be given and suppose $\mathcal{A}$ is a collection of open sets in X which cover $[a, b]$. Define the set $S \subseteq [a, b]$ by

$$S = \{x \in X : [a, x] \text{ is covered by a finite subcollection of } \mathcal{A}\}.$$

As $\mathcal{A}$ covers $[a, b]$ there is some open set A containing a. Then there exists x such that $[a, x) \subseteq A$. If x is the immediate successor of a then there is some set $A' \in \mathcal{A}$ such that $x \in A'$, hence $x \in S$ as $[a, x] = \{a, x\}$ is finitely covered. Otherwise, there exists some y such that $a < y < x$. As $[a, x)$ is convex we have $[a, y] \subseteq [a, x) \subseteq A$. Thus $y \in S$.

In any case, S is non-empty and bounded above by b. Let $c = \sup S$. We first show that $c \in S$. By construction of S, if $x < c$ then $x \in S$. As $\mathcal{A}$ is a cover of $[a, b]$ there exists and open set $A_c \in \mathcal{A}$ such that $c \in A_c$ and by openness we have some open interval (d, e) containing c. As $d < c$ we have $d \in S$. If d is the immediate predecessor of c we apply the same argument as above and conclude $c \in S$. Otherwise there is some $y \in S$ such that $d < y < c$ and again by convexity of the open interval (d, e) we have $[y, c] \subseteq (d, e) \subseteq A_c$. Possibly adding A_c to the finite covering of $[a, y]$ we have $c \in S$.

We claim that $c = b$ and verify by way of contradiction. Suppose $c < b$. As before we have an open interval (d, e) containing c. We can assume $e \leq b$. If e is the

immediate successor of c we can again extend the finite cover to include e. Otherwise there exists z such that $c < z < e$ and we extend the finite cover to $[a, z]$. In either case there exists an element of S greater than c contradicting the definition of c. Thus we conclude that $c = b$.

We now have $b \in S$ and by definition of S the interval $[a, b]$ has a finite subcollection of $\mathcal{A}$ as a covering. As $\mathcal{A}$ was arbitrary, we have determined that the closed interval $[a, b]$ is a compact subspace of X. □

Corollary 12.20. *Every closed interval of $\mathbb{R}$ is compact.*

We have some nice immediate consequences.

Example 12.21. Every path $f : [0, 1] \to X$ is compact.

Example 12.22. The unit sphere S^n in $\mathbb{R}^{n+1}$ is compact. For this we need a simple extension of the homework to show $S^n \cong \Sigma S^{n-1}$ and proceed by induction. For the base case we have S^1 is the image of the interval $[0, 1]$ under the continuous function $f(t) = (\cos t, \sin t)$ and is thus compact. Assuming S^{n-1} is compact, we extend the continuous function found in the homework to a continuous function $g : S^{n-1} \times [0, 1] \to S^n$. By Theorem 12.16 and Corollary 12.20, $S^{n-1} \times [0, 1]$ is compact and by Theorem 12.12 S^n is compact.

Theorem 12.23. *A subspace A of $\mathbb{R}^n$ is compact if and only if it is closed and bounded in the Euclidean metric d or the square metric ρ.*

Proof. Recall that d and ρ generate the standard topology on $\mathbb{R}^n$ by Theorem 9.8. If $x \in \mathbb{R}^n$ and all $\varepsilon > 0$ we have $B_\rho(x, \varepsilon) \subset B_d(x, \varepsilon), \subset B_\rho(x, \sqrt{n}\varepsilon)$, then a subset $A \subset \mathbb{R}^n$ is bounded under d if and only if it is bounded under ρ. Thus we consider only the metric ρ.

($\Rightarrow$) Suppose $A \subset \mathbb{R}^n$ is compact. As $\mathbb{R}^n$ is Hausdorff, A is closed by Theorem 12.8. Consider the collection

$$\mathcal{B} = \{B_\rho(\mathbf{0}, m) : m \in \mathbb{Z}_+\}.$$

As $\mathbb{R}^n = \bigcup_{m \in \mathbb{Z}_+} B_\rho(\mathbf{0}, m)$, $\mathcal{B}$ is a covering of A by open subsets of $\mathbb{R}^n$. As A was assumed to be compact there is a finite subcollection $\{B_\rho(\mathbf{0}, m_i)\}_{1 \le i \le n}$ which covers A. The set $\{m_i\}_{1 \le i \le n}$ is a finite set and thus has a maximal element M. As $m_i \le M$ implies $B_\rho(\mathbf{0}, m_i) \subseteq B_\rho(\mathbf{0}, M)$ we have $A \subseteq B_\rho(\mathbf{0}, M)$. Hence A is bounded.

($\Leftarrow$) Suppose A is closed and bounded under ρ. Then there exists M such that $\rho(x, y) \le N$ for all $x, y \in A$. Fix a point $a \in A$ and let $N = \rho(a, \mathbf{0})$. By the triangle inequality, $\rho(x, \mathbf{0}) \le M + N$ for all $x \in A$. Thus

$$A \subseteq \bar{B}_\rho(\mathbf{0}, M + N) = \overbrace{[-M - N, M + N] \times \cdots \times [-M - N, M + N]}^{n \text{ times}}.$$

Then by Corollary 12.20 and Theorem 12.16 $\bar{B}_\rho(\mathbf{0}, M + N)$ is compact. By Theorem 12.7, being a closed subspace of a compact space A is in turn compact. □

Example 12.24. The closed unit balls $\bar{B}_d(0, 1)$ and $\bar{B}_\rho(0, 1)$ are compact as they are closed and bounded.

We can now generalize the Extreme Value Theorem from calculus to functions from arbitrary compact spaces. Like the generalization of the Intermediate Value Theorem, the codomain will be generalized to sets with an order topology.

Theorem 12.25 (Extreme Value Theorem)**.** *Let X be a compact space and Y a set with a strict total order endowed with the order topology. If $f : X \to Y$ is a continuous function then there exists points $c, d \in X$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in X$.*

Proof. As f is continuous $f(X)$ is continuous. If $f(X)$ has no largest element then the collection

$$\{(-\infty, y) : y \in f(X)\}$$

forms an open cover of $f(X)$. By compactness there is some finite subcollection $\{(-\infty, y_1), (-\infty, y_2), \ldots, (-\infty, y_n)\}$ which covers $f(X)$. Let $y = \max_{1 \leq i \leq n}\{y_i\} \in f(X)$. Then y is not in $\bigcup_{i=1}^{n}(-\infty, y_i)$ and hence the subcollection does not cover $f(X)$, a contradiction. Thus $f(X)$ contains a largest element, M. A similar argument with $\{(y, \infty) : y \in f(X)\}$ shows $f(X)$ contains a smallest element m.

Since $m, M \in f(X)$ there is some $c, d \in X$ such that $f(c) = m$ and $f(d) = M$. By definition of m and M we have $f(c) \leq f(x) \leq f(d)$ for all $x \in X$. □

We also have a suitable generalization of the Uniform Continuity Theorem, however we do need a technical lemma first.

Definition 12.26. Let (X, d) be a metric space and A a non-empty subset of X. For a point $x \in X$, *distance from x to A* is defined as

$$d(x, A) = \inf\{d(x, a) : a \in A\}.$$

The *diameter* of A is defined as

$$\text{diam}A = \sup\{d(a_1, a_2) : a_1, a_2 \in A\}.$$

Lemma 12.27. *For a fixed subset A, the function*

$$d(\cdot, A) : X \to \mathbb{R}$$

is continuous

Proof. Let $x \in X$ be a point and let $\varepsilon > 0$. Suppose for $y \in Y$ that $d(x, y) < \varepsilon$. For $a \in A$ we have by the triangle inequality

$$d(x, a) \leq d(y, a) + d(x, y)$$
$$d(y, a) \leq d(x, a) + d(x, y).$$

As d is a metric and the above holds for all $a \in A$ we have

$$d(x, A) - d(y, A) \leq d(x, a_0) - \inf\{d(y, a) : a \in A\} \leq d(x, y)$$
$$d(y, A) - d(x, A) \leq d(y, a_0) - \inf\{d(x, a) : a \in A\} \leq d(x, y)$$

Thus $|d(x, A) - d(y, A)| \leq d(x, y) < \varepsilon$ and we conclude that $d(\cdot, A)$ is continuous. □

Lemma 12.28 (The Lebesgue Number Lemma)**.** *Let (X, d) be a compact metric space and $\mathcal{A}$ an open covering of X. There exists some $\delta > 0$ such that for each subset of X having diameter less than δ, there exists some $A \in \mathcal{A}$ containing it. We refer to δ as the* ***Lebesgue number*** *of $\mathcal{A}$.*

Proof. Let $\mathcal{A}$ be an open cover of X. If $X \in A$ then every real number is a Lebesgue number. Assume that $X \notin A$.

As X is compact there is a finite subcollection $\{A_1, \ldots, A_n\}$ of $\mathcal{A}$ that covers X. Define $C_i = X - A_i$ and let $f : X \to \mathbb{R}$ be the function give by

$$f(x) = \frac{1}{n} \sum_{i=1}^{n} d(x, C_i).$$

For $x \in X$ there is some A_i such that $x \in A_i$. Since A_i is open there is some ε such that $B_d(x, \varepsilon) \subseteq A_i$. Then $d(x, C_i) \geq \varepsilon$. Thus $f(x) \geq \frac{\varepsilon}{n}$. In particular we have that f is strictly positive for all $x \in X$.

As X is compact and f is continuous, $f(X)$ has a smallest element δ. Let B be a subset of X of diameter less than δ. Choose a point x_0 in B; we have $B \subseteq B_d(x_0, \delta)$. As

$$\delta \leq f(x_0) = \frac{1}{n} \sum_{i=1}^{n} d(x_0, C_i) \leq n \frac{1}{n} \max_{1 \leq i \leq n} \{d(x_0, C_i)\}.$$

Let C_k be such that $d(x_0, C_k) = \max_{1 \leq i \leq n} \{d(x_0, C_i)\}$. Then $B_d(x_0, \delta) \subseteq A_k = X - C_k$. As B was an arbitrary set of diameter δ, δ is a Lebesgue number for $\mathcal{A}$. □

Definition 12.29. A function $f : (X, d_X) \to (Y, d_Y)$ between metric spaces is said to be *uniformly continuous* if given $\varepsilon > 0$ there exists a $\delta > 0$ such that for every pair of points $x_0, x_1 \in X$ we have

$$d_X(x_0, x_1) < \delta \Rightarrow d_Y(f(x_0), f(x_1)) < \varepsilon.$$

Theorem 12.30 (Uniform Continuity Theorem)**.** *Let $f : (X, d_X) \to (Y, d_Y)$ be a function from a compact metric space (X, d_X) to a metric space (Y, d_Y). If f is continuous, then f is uniformly continuous.*

Proof. Let $\varepsilon > 0$ be given. The collection $\{B_{d_Y}(y, \frac{\varepsilon}{2})\}$ is an open cover of Y. Since f is continuous the collection $\mathcal{A} = \{f^{-1}(B_{d_Y}(y, \frac{\varepsilon}{2}))\}$ is an open cover of X. Let δ be a Lebesgue number for $\mathcal{A}$.

If $d_X(x_0, x_1) < \delta$ the set $\{x_0, x_1\}$ is of diameter less than δ. By the Lebesgue Number Lemma there is some element $A \in \mathcal{A}$ such that $\{x_0, x_1\} \subseteq A$. As each element of $\mathcal{A}$ is of the form $A = f^{-1}(B_{d_Y}(y, \frac{\varepsilon}{2}))$, we have

$$\begin{aligned} f(\{x_0, x_1\}) &\subseteq f(f^{-1}(B_{d_Y}(y, \frac{\varepsilon}{2}))) \\ &\subseteq B_{d_Y}(y, \frac{\varepsilon}{2}) \end{aligned}$$

By the triangle inequality $d_Y(f(x_0), f(x_1)) < \varepsilon$. We conclude that f is uniformly continuous. □

We end this section with an interesting proof that $\mathbb{R}$ is uncountable.

Definition 12.31. For a topological space X, we call a point $x \in X$ an *isolated point* if the one-point set $\{x\}$ is open in X.

Theorem 12.32. *Let X be a non-empty compact Hausdorff space. If X has no isolated points, then X is uncountable.*

Proof. Suppose X has no isolated points and let $f : \mathbb{Z}_+ \to X$ be a given function. We will show f is not surjective and by generality of the function f conclude that X is uncountable.

Define $x_n = f(n)$. Let V_0 be a non-empty open subset of X. There exists some $y \in V_0$ such that $x_0 \neq y$. For if $x_0 \in V_0$ we know $V_0 \neq \{x_1\}$ as X has no isolated points. As X is Hausdorff there exists open sets W_1, W_y such that $x_1 \in W_1, y \in W_y$ and $W_1 \cap W_y = \emptyset$. Then $V_1 = W_2 \cap V_0$ is such that $y \in V_1$ and $x_1 \notin \bar{V}_1$ (x_1 cannot be a limit point of V_1 as $W_1 \cap V_1 = \emptyset$).

Applying the above, to V_{n-1} we get a set V_n such that $x_n \notin \bar{V}_n$. Then

$$\bar{V}_1 \supseteq \bar{V}_2 \supseteq \cdots$$

is a collection of non-empty closed sets in X. By Theorem 12.18 there is a point $x \in \bigcap_{i=1}^{\infty} \bar{V}_i$. As $x_n \notin \bar{V}_n$ and $x \in V_n$ we conclude that $x \neq x_n = f(n)$ for any $n \in \mathbb{Z}_+$. Thus f is not surjective and we conclude the proof. □

Corollary 12.33. *Every closed interval in $\mathbb{R}$ is uncountable; $\mathbb{R}$ is uncountable.*

Proof. As $\mathbb{R}$ is Hausdorff, $[a, b]$ is Hausdorff. Thus singletons $\{x\}$ are closed. By Corollary 11.20 $[a, b]$ is connected, hence $\{x\}$ cannot be open. Thus $[a, b]$ is compact (Corollary 12.20) Hausdorff and has no isolated points. By Theorem 12.32, $[a, b]$ is uncountable. The inclusion map $\iota : [a, b] \to \mathbb{R}$ given by $\iota(x) = x$ implies $\mathbb{R}$ is uncountable. □